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INTRODUCTION TO FINITE ELEMENT METHOD

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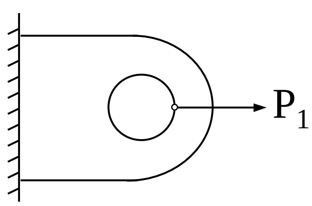
MEET THE FINITE ELEMENT METHOD

SOLVING A PROBLEM

Key ingredients that lead to various solutions:

1. Differential equations
2. Boundary conditions
3. Geometries

LUG PROBLEM (1)



Differential equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

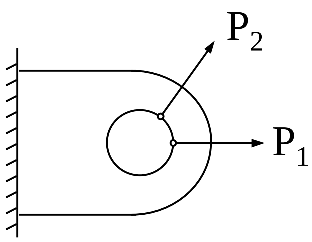
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

Boundary conditions:__Fixed along left edge with applied force P_1

Geometry:_Plate with straight/round edges and circular cutout

Result:_____Deformation and stress solution

LUG PROBLEM (2)



Differential equations:

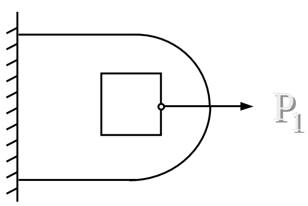
(Same as before)

Boundary conditions:__With additional force P_2

Geometry:_(Same as before)

Result:_____“Another” deformation and stress solution

LUG PROBLEM (3)



Differential equations:

(Same as before)

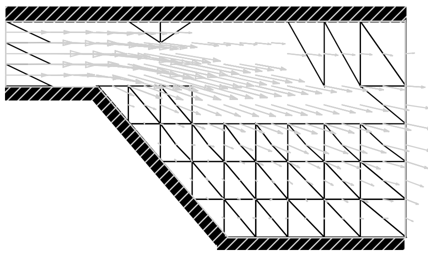
Boundary conditions:__(Same as the original one)

Geometry:_The cutout is “square”

Result:_____“Another” deformation and stress solution

FLOW IN CHANNEL (1)

Differential equations:



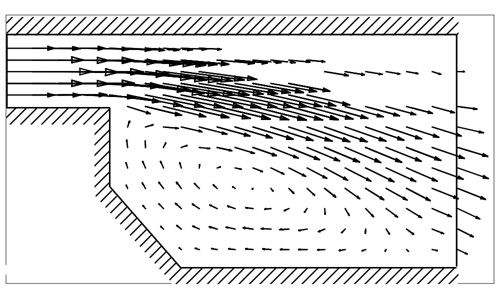
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\partial p}{\partial x} = \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)$$

$$\mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{\partial p}{\partial y} = \rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)$$

Boundary conditions:__As shown in figure
Geometry:____________As shown in figure
Result:_______________A set of u-v-p solution

FLOW IN CHANNEL (2)



Differential equations:

(Same Navier-Stokes Eqs.)

Boundary conditions:____(Same as before)
Geometry:__More fluid area on lower-left region
Result:______“Another” set of u-v-p solution

WAYS TO SOLVE THESE PROBLEMS

1. Solve for exact or analytical solutions:
   - 1.1 Separation of Variables
   - 1.2 Laplace Transform

     Etc. But solutions are limited.
2. Solve for approximate solutions:
   - 2.1 Finite Difference Method
   - 2.2 Finite Element Method

     Etc.

WHAT IS THE FINITE ELEMENT METHOD?

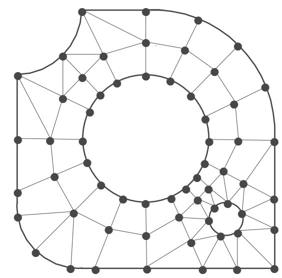
Very helpful to understand the "finite difference method" first:

- Both are numerical methods for obtaining approximate solutions to differential equations
- Both have several common characteristics
- Will give definitions of both methods later

PLATE WITH COMPLEX SHAPE

Plate Geometry

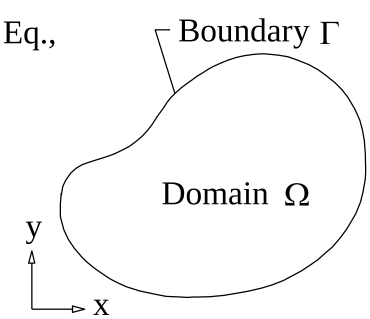
Finite Difference Model



Finite Element Model

FINITE DIFFERENCE METHOD

Want approx. solution to the Laplace Eq.,



$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

where,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

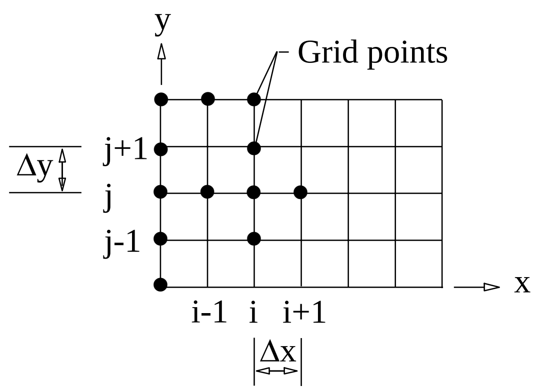
$\phi = \phi(x, y)$ = Unknown variable in domain Ω
(e.g. temperature T = T(x, y))

f = f(x,y) = known along boundary Γ
(e.g. specified T = 30 ^{o}C along boundary)

Want $\phi = \phi(x, y)$ = ?

GENERAL PROCEDURE IN FINITE DIFFERENCE METHOD

Step 1: Lay out the mesh (eg. we have a rectangular region in x-y plane)



GENERAL PROCEDURE IN FINITE DIFFERENCE METHOD

Step 2: Write PDE in terms of grid unknowns.

PDE: $$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

Apply Taylor series expansion. Since

$$\phi_{i+1} = \phi_i + \left.\frac{\partial\phi}{\partial x}\right|_i \Delta x + \frac{1}{2!}\left.\frac{\partial^2\phi}{\partial x^2}\right|_i (\Delta x)^2 + \frac{1}{3!}\left.\frac{\partial^3\phi}{\partial x^3}\right|_i (\Delta x)^3 + \frac{1}{4!}\left.\frac{\partial^4\phi}{\partial x^4}\right|_i (\Delta x)^4 + \ldots$$

and

$$\phi_{i-1} = \phi_i - \left.\frac{\partial\phi}{\partial x}\right|_i \Delta x + \frac{1}{2!}\left.\frac{\partial^2\phi}{\partial x^2}\right|_i (\Delta x)^2 - \frac{1}{3!}\left.\frac{\partial^3\phi}{\partial x^3}\right|_i (\Delta x)^3 + \frac{1}{4!}\left.\frac{\partial^4\phi}{\partial x^4}\right|_i (\Delta x)^4 - \ldots$$

Combine these two eqs. to obtain,

GENERAL PROCEDURE IN FINITE DIFFERENCE METHOD

$$\phi_{i+1} + \phi_{i-1} = 2\phi_i + \frac{2}{2!}\frac{\partial^2\phi}{\partial x^2}\bigg|_i (\Delta x)^2 + \frac{2}{4!}\frac{\partial^4\phi}{\partial x^4}\bigg|_i (\Delta x)^4 + \ldots$$

Then, neglect higher order derivative terms to yield,

$$\frac{\partial^2\phi}{\partial x^2} \cong \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{(\Delta x)^2}$$

Similary, in y-direction,

$$\frac{\partial^2\phi}{\partial y^2} \cong \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{(\Delta y)^2}$$

Substitute these into Laplace Eq. to get,

GENERAL PROCEDURE IN FD METHOD

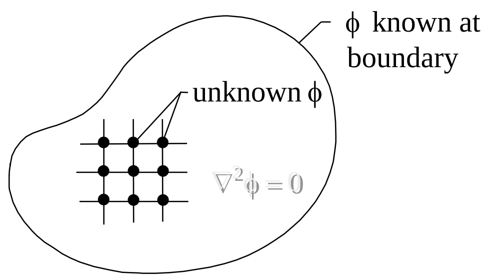
$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \quad \Rightarrow \quad \phi_{i+1} + \phi_{i-1} + \phi_{j+1} + \phi_{j-1} \boxed{-} 4\phi_i = 0$$

$$\Downarrow$$

“Stencil form” $\quad \begin{array}{ccccc} & & 1 & & \\ & & | & & \\ 1 & — & -4 & — & 1 \\ & & | & & \\ & & 1 & & \end{array} \quad = \quad 0$

Then, to solve a problem, we apply this “approx. PDE” at all nodes which are unknowns, i.e.,

GENERAL PROCEDURE IN FINITE DIFFERENCE METHOD

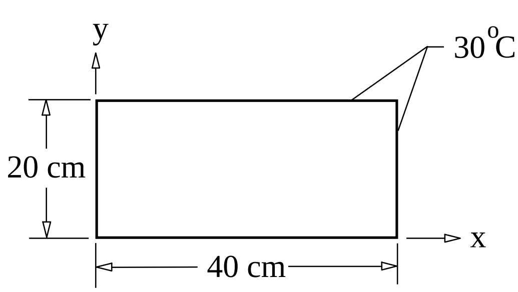


Step 3: Write a set of simultaneous eqs. for all the nodes which are unknown.

Step 4: Solve for nodal unknowns.

FINITE DIFFERENCE EXAMPLE

Steady-state heat conduction in a rectangular plate.



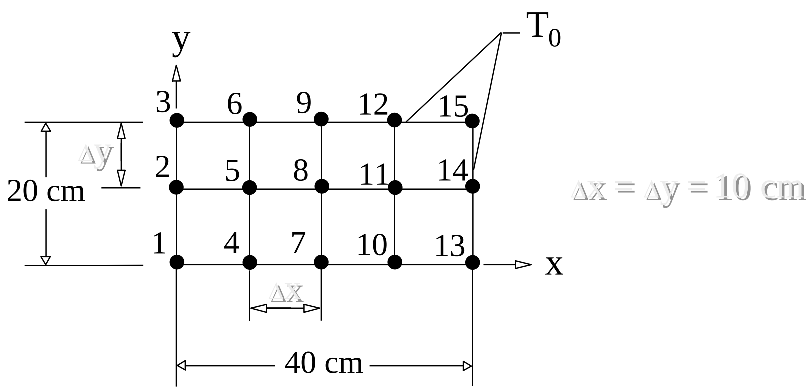
Laplace equation,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Determine temperature distribution by the finite difference method.

FINITE DIFFERENCE EXAMPLE

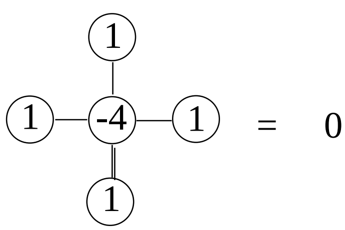
Step 1: Lay out the mesh:



Here, unknowns are T_5, T_8 and T_{11} (only 3 unknowns).

FINITE DIFFERENCE EXAMPLE

Step 2: Apply the "stencil form" at all the nodes which are unknown. Here, we apply at nodes 5, 8 and 11.



Applying at node 5 to get,

$$T_2 + T_6 + T_8 + T_4 - 4T_5 = 0$$

but $T_2 = T_6 = T_4$ which is equal to T_0, then

$$4T_5 - T_8 = 3T_0$$

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FINITE DIFFERENCE EXAMPLE

Similary, applying at node 8 to get,

$$T_5 + T_9 + T_{11} + T_7 - 4T_8 = 0$$

but $T_9 = T_7 = T_0$, then

$$-T_5 + 4T_8 - T_{11} = 2T_0$$

Also, applying at node 11 to get,

$$T_8 + T_{12} + T_{14} + T_{10} - 4T_{11} = 0$$

but $T_{12} = T_{14} = T_{10} = T_0$, then

$$-T_8 + 4T_{11} = 3T_0$$

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FINITE DIFFERENCE EXAMPLE

Step 3: These 3 eqs. lead to a set of simultaneous eqs.,

$$\begin{aligned} 4T_5 - T_8 \quad\quad &= 3T_0 \\ -T_5 + 4T_8 - T_{11} &= 2T_0 \\ - T_8 + 4T_{11} &= 3T_0 \end{aligned}$$

Or, in matrix form,

$$\underbrace{\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}}_{\substack{[K] \\ (3x3)}} \underbrace{\begin{Bmatrix} T_5 \\ T_8 \\ T_{11} \end{Bmatrix}}_{\substack{\{T\} \\ (3x1)}} = \underbrace{\begin{Bmatrix} 3T_0 \\ 2T_0 \\ 3T_0 \end{Bmatrix}}_{\substack{\{Q\} \\ (3x1)}}$$

FINITE DIFFERENCE EXAMPLE

In short, $$\underset{(3x3)}{[K]}\ \underset{(3x1)}{\{T\}} = \underset{(3x1)}{\{Q\}}$$

Here $[K]$ is called "Conduction matrix"

$\{T\}$ is called "Vector of nodal unknowns"

$\{Q\}$ is called "Load vector"

Note that the Conduction matrix $[K]$ has the following characteristics,

1. Diagonal terms dominated
2. Positive values on diagonal terms
3. Symmetric matrix, $[K] = [K]^T$

FINITE DIFFERENCE EXAMPLE

and 4. Banded

$$\begin{bmatrix} x & x & & & 0 \\ x & x & x & & \\ & & & & \\ & & x & x & x \\ 0 & & & x & x \end{bmatrix}$$

x = nonzero terms

which is helpful in programming for reducing computer memory. In addition, if $[K]$ is symmetric, then only upper half can be stored resulting in further memory reduction,

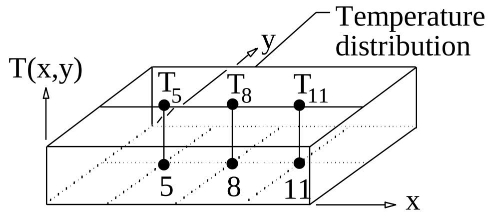
$$\begin{bmatrix} x & x & & 0 \\ & x & x & \\ & & & \\ & & x & x \\ & & & x \end{bmatrix}$$

FINITE DIFFERENCE EXAMPLE

Step 4: Solve for nodal unknowns to obtain,

$$T_5 = T_8 = T_{11} = T_0 = 30\,^{o}C$$

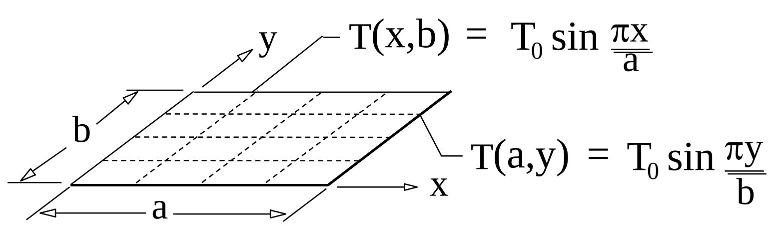
which are equal to the edge temperature.



Note that these nodal unknowns are exact solution. Why we obtain exact solution eventhough we used truncated Taylor series?

FINITE DIFFERENCE EXAMPLE

This is because temperature distribution for this problem is uniform. Thus, higher order derivative terms in Taylor series are automatically zero. Finite difference won't give exact solution for the problem below,



However, solution accuracy increases if the mesh is refined.

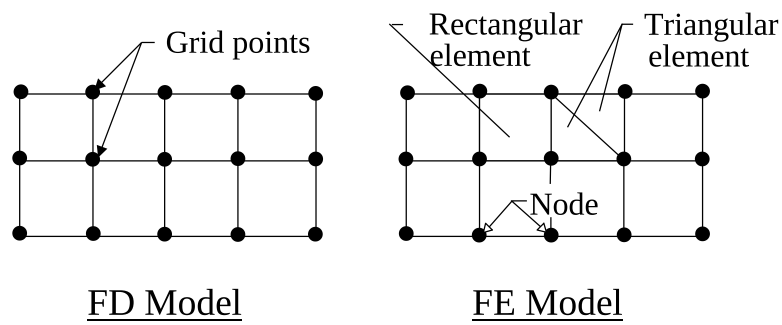
FINITE DIFFERENCE VS FINITE ELEMENT

The Finite Difference Method is a technique to approximately solve ordinary and partial differential equations. In this method, the derivatives appear in the differential equations are replaced by algebraic approximation in term of values of the dependent variables at grid points throughout the region.

FINITE DIFFERENCE VS FINITE ELEMENT

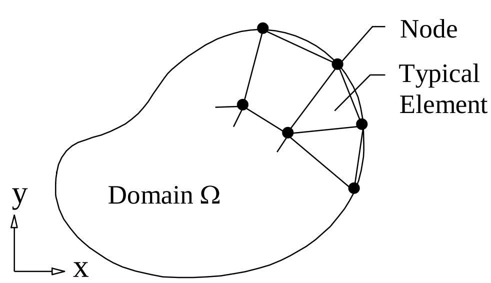
The Finite Element Method is a method to solve a problem by physically dividing the region into small segments known as elements. The elements are connected at discrete points called nodes at which the dependent variables are determined.

MODELLING OF HEAT CONDUCTION IN PLATE



GENERAL PROCEDURE IN FINITE ELEMENT METHOD

Step 1: Discretize the solution domain into a number of elements:



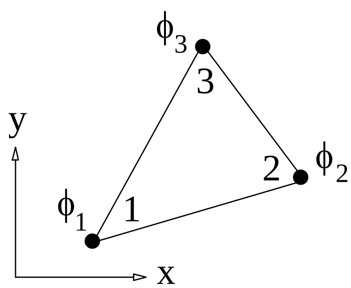
Domain could be for:

- Elasticity problem
- Thermal problem
- Fluid problem

GENERAL PROCEDURE IN FINITE ELEMENT METHOD

<u>Step 2</u>: Select "Element interpolation functions", e.g., for triangular element:

ϕ_i = nodal unknowns, i = 1, 2, 3



<u>Problem</u>	<u>ϕ_i</u>
Elasticity	Displacements u, v
Thermal	Temperature T
Fluid	Velocities u, v & Pressure p

GENERAL PROCEDURE IN FE METHOD

Distribution of element variable is expressed in terms of nodal unknowns as,

$$\phi(x,y) = N_1(x,y)\,\phi_1 + N_2(x,y)\,\phi_2 + N_3(x,y)\,\phi_3$$

Where $N_i(x,y)$, i = 1, 2, 3 are the element interpolation functions. or, in matrix form,

$$\phi(x,y) = \lfloor N_1 \quad N_2 \quad N_3 \rfloor \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \underset{(1\text{x}3)}{\lfloor N \rfloor}\ \underset{(3\text{x}1)}{\{\phi\}}$$

where $\lfloor N \rfloor$ is element interpolation function matrix, and $\{\phi\}$ is vector of element nodal unknowns.

GENERAL PROCEDURE IN FE METHOD

Step 3: Set up element equations for each element.

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}_e \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}_e = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}_e$$

Or, $$[K]_e \{\phi\}_e = \{F\}_e$$

These element equations can be derived using,

(1) Direct approach

(2) Variational approach

(3) Method of Weighted Residuals

GENERAL PROCEDURE IN FINITE ELEMENT METHOD

Step 4: Assemble all element eqs. to obtain system of simultaneous eqs.,

$$\sum(\text{element equations}) \Rightarrow [K]_{sys} \{\phi\}_{sys} = \{F\}_{sys}$$

Step 5: Apply boundary conditions and solve for nodal unknowns $\{\phi\}_{sys}$, e.g., solve for;

Nodal displacements	(Elasticity problem)
Nodal temperatures	(Thermal problem)
Nodal flow velocities	(Fluid problem)

GENERAL PROCEDURE IN FINITE ELEMENT METHOD

Step 6: Compute any other quantities of interest. For examples,

Problem	Knowing	Can further compute
Elasticity	Displacements	Stresses
Thermal	Temperatures	Heat fluxes
Fluid	Velocities	Flow rates

GENERAL PROCEDURE IN FINITE ELEMENT METHOD

Note that step 3, derivation of finite element eqs., is the heart of the finite element method.

- Typical structural FE program can be modified to solve heat transfer problem by replacing the finite element eqs.
- Thus, it is important to understand how to derive the finite element eqs.

DERIVATION OF FINITE ELEMENT EQUATIONS

Three Approaches:

1. Direct Approach
   - Simple but has limitation
2. Variational Approach
   - Originated from structural area
3. Method of Weighted Residuals
   - More general approach applicable to other fields