THE DIRECT APPROACH

THE DIRECT APPROACH

Objective: Derive "finite element equations" from physical understanding.

Will set up element eqs. for:

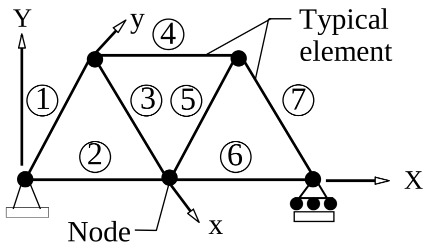
(a) Truss problem (Rod or Spring element)

(b) Thermal problem (1-D heat conduction)

(c) Fluid problem (Flow in pipe)

ELEMENT EQS FOR TRUSS PROBLEM

Consider 2-D truss problem,



Finite element model consists of:

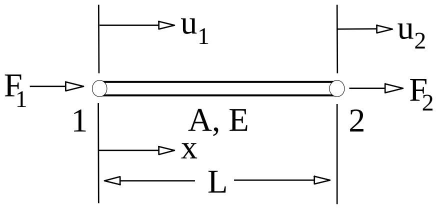
7 Elements

5 Nodes

X-Y is the global coordinates of the entire system

x- y is the local coordinates of element no. ③

DERIVATION OF ELEMENT EQUATIONS



u_1 & u_2 = Nodal displacements

F_1 & F_2 = Nodal forces

A = Cross-sectional area

E = Modulus of elasticity

From Hooke's Law: $\sigma = E\varepsilon$

Sub. for stress and strain: $\frac{F_2}{A} = E\left(\frac{u_2 - u_1}{L}\right)$

or, $-\frac{AE}{L}(u_1 - u_2) = F_2$

TRUSS ELEMENT

For equilibrium: $\sum F_x = 0$

$$F_1 + F_2 = 0$$

$$F_1 = -F_2$$

Or, $F_1 = \frac{AE}{L}\left(u_1 - u_2\right)$

The two equations are,

$$\frac{AE}{L}\left(u_1 - u_2\right) = F_1$$

$$-\frac{AE}{L}\left(u_1 - u_2\right) = F_2$$

TRUSS ELEMENT

which can be written in matrix form (truss element equations) as,

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

or, in short, $[K]\{u\} = \{F\}$

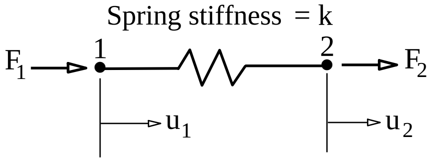
where $[K]$ = Element stiffness matrix

$\{u\}$ = Vector of element nodal displacements

$\{F\}$ = Vector of element nodal forces

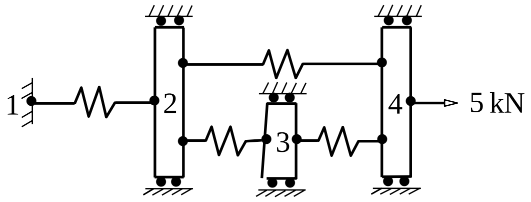
SPRING ELEMENT

Similary, the finite element equations for a spring element can be derived,



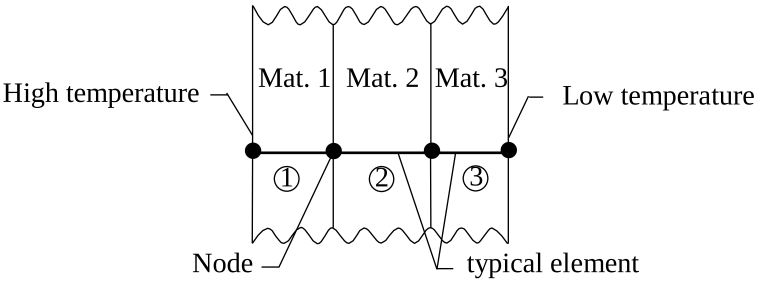
$$k\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}=\begin{Bmatrix}F_1\\ F_2\end{Bmatrix}$$

These FE eqs. can be used to solve problems like,



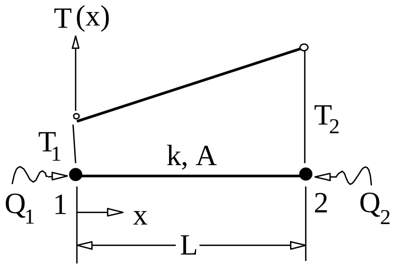
ELEMENT EQS FOR THERMAL PROBLEM

Consider 1-D steady-state heat conduction in slab with different materials,



Again, want to derive finite element equations.

DERIVATION OF ELEMENT EQUATIONS



T_1 & T_2 = Nodal temperatures

Q_1 & Q_2 = Nodal fluxes

k = Thermal conductivity

A = Conduction area

From Fourier's Law: $Q = -k\,A\,\dfrac{\partial T}{\partial x}$

Thus, $Q_1 = -k\,A\,\dfrac{T_2 - T_1}{L} = \dfrac{kA}{L}\left(T_1 - T_2\right)$

CONDUCTION ELEMENT EQS

Energy balance: $Q_1 + Q_2 = 0$

$$Q_2 = -Q_1$$

or, $Q_2 = \dfrac{kA}{L}\left(-T_1 + T_2\right)$

The two equations are,

$$\frac{kA}{L}\left(T_1 - T_2\right) = Q_1$$

$$\frac{kA}{L}\left(-T_1 + T_2\right) = Q_2$$

CONDUCTION ELEMENT EQS

which can be written in matrix form as,

$$\frac{kA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = \begin{Bmatrix}Q_1\\ Q_2\end{Bmatrix}$$

or, in short, $$[K]\{T\} = \{Q\}$$

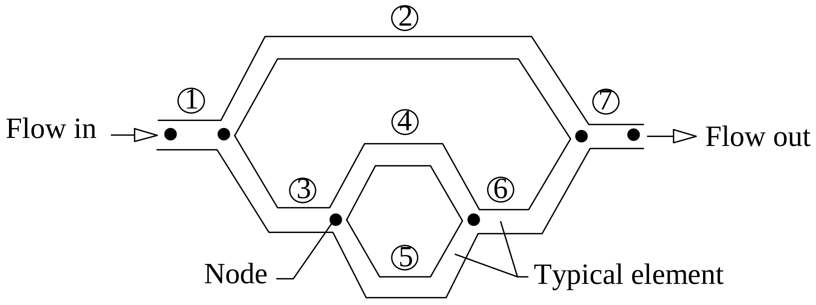
where $[K]$ = Element conduction matrix

$\{T\}$ = Vector of element nodal temperatures

$\{Q\}$ = Vector of element nodal heat fluxes

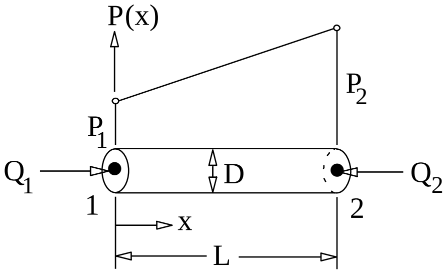
ELEMENT EQS FOR FLUID PROBLEM

Consider fully developed, laminar, incompressible flow in a pipe network as shown:



Again, want to derive <u>finite element equations</u>.

DERIVATION OF ELEMENT EQUATIONS



Flow in circular pipe:

P_1 & P_2 = Nodal pressures

Q_1 & Q_2 = Nodal flow rates

μ = Fluid viscosity

D = Pipe diameter

Flow rate:

$$Q = -\frac{\pi D^4}{128\mu}\frac{\partial P}{\partial x}$$

Thus,

$$Q_1 = -\frac{\pi D^4}{128\mu}\frac{P_2 - P_1}{L} = \frac{\pi D^4}{128\mu L}\left(P_1 - P_2\right)$$

FLUID ELEMENT EQUATIONS

Conservation of mass:

$$Q_1 + Q_2 = 0$$

$$Q_2 = -Q_1$$

or,

$$Q_2 = \frac{\pi D^4}{128\mu L}\left(-P_1 + P_2\right)$$

The two equations are,

$$\frac{\pi D^4}{128\mu L}\left(P_1 - P_2\right) = Q_1$$

$$\frac{\pi D^4}{128\mu L}\left(-P_1 + P_2\right) = Q_2$$

FLUID ELEMENT EQUATIONS

which can be written in matrix form as,

$$\frac{\pi D^4}{128\mu L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

or, in short, $[K]\{P\} = \{Q\}$

where $[K]$ = Element fluidity matrix

$\{P\}$ = Vector of element nodal pressures

$\{Q\}$ = Vector of element nodal flow rates

FINITE ELEMENT EQUATIONS

In conclusion, element equations obtained from these 3 problems are in the same form, i.e.,

$$[K]\{u\} = \{F\}$$

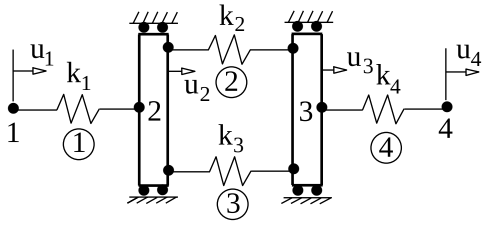
where $[K]$ = Element stiffness matrix

$\{u\}$ = Vector of element nodal unknowns

$\{F\}$ = Vector of element nodal loads

SYSTEM EQUATIONS

Example Set up system equations for the spring system,



Since there are 4 unknowns, thus there must be 4 eqs.,

$$\underset{(4x4)}{\begin{bmatrix} \bar{k}_{11} & \bar{k}_{12} & \bar{k}_{13} & \bar{k}_{14} \\ \bar{k}_{21} & \bar{k}_{22} & \bar{k}_{23} & \bar{k}_{24} \\ \bar{k}_{31} & \bar{k}_{32} & \bar{k}_{33} & \bar{k}_{34} \\ \bar{k}_{41} & \bar{k}_{42} & \bar{k}_{43} & \bar{k}_{44} \end{bmatrix}} \underset{(4x1)}{\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}} = \underset{(4x1)}{\begin{Bmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \\ \bar{f}_4 \end{Bmatrix}}$$

Or, in short,

$$[K]_{sys} \{u\}_{sys} = \{F\}_{sys}$$

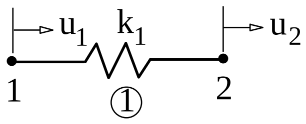
SYSTEM STIFFNESS MATRIX

Basic idea:

$$[K]_{sys} = \sum_{elements} [K]_e$$

Consider element ① connected between nodes 1 & 2:

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$



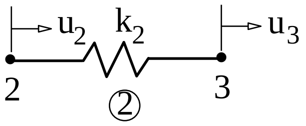
which can be written in form of system eqs. (4 eqs.) as,

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ 0 \\ 0 \end{Bmatrix} \qquad (A)$$

SYSTEM STIFFNESS MATRIX

Similary, for element ② connected between nodes 2 & 3:

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$



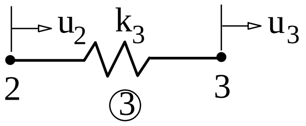
which can be written in form of system eqs. (4 eqs.) as,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_2 \\ F_3 \\ 0 \end{Bmatrix} \qquad \text{(B)}$$

SYSTEM STIFFNESS MATRIX

And, for element ③ connected between nodes 2 & 3:

$$\begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$



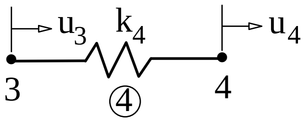
which can be written in form of system eqs. (4 eqs.) as,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_2 \\ F_3 \\ 0 \end{Bmatrix} \qquad \text{(C)}$$

SYSTEM STIFFNESS MATRIX

And, the last element ④ connected between nodes 3 & 4:

$$\begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$



which can be written in form of system eqs. (4 eqs.) as,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F_3 \\ F_4 \end{Bmatrix} \qquad \text{(D)}$$

SYSTEM STIFFNESS MATRIX

Set of eqs, (A), (B), (C), (D) can be combined leading to the system stiffness matrix as,

$$[K]_{sys} = \sum [K]_e$$

$$= \begin{array}{c} \begin{array}{cccc} (1) & (2) & (3) & (4) \end{array} \\ \left[\begin{array}{c|c|c|c} k_1 & -k_1 & & \\ \hline -k_1 & k_1+k_2+k_3 & -k_2-k_3 & \\ \hline & -k_2-k_3 & k_2+k_3+k_4 & -k_4 \\ \hline & & -k_4 & k_4 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \end{array} \end{array}$$

where the numbers (1), (2), (3), (4) denote the corresponding node numbers.

SYSTEM STIFFNESS MATRIX

Note that, the system stiffness matrix can be obtained easier by simply assigning the node numbers on the element stiffness matrices (i.e. the numbers (1), (2), (3), (4) below),

$$[K]_{\text{element①}} = \begin{array}{cc} \begin{array}{cc} (1) & (2) \end{array} & \\ \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} & \begin{array}{c} (1) \\ (2) \end{array} \end{array} \qquad [K]_{\text{element②}} = \begin{array}{cc} \begin{array}{cc} (2) & (3) \end{array} & \\ \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} & \begin{array}{c} (2) \\ (3) \end{array} \end{array}$$

$$[K]_{\text{element③}} = \begin{array}{cc} \begin{array}{cc} (2) & (3) \end{array} & \\ \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} & \begin{array}{c} (2) \\ (3) \end{array} \end{array} \qquad [K]_{\text{element④}} = \begin{array}{cc} \begin{array}{cc} (3) & (4) \end{array} & \\ \begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix} & \begin{array}{c} (3) \\ (4) \end{array} \end{array}$$

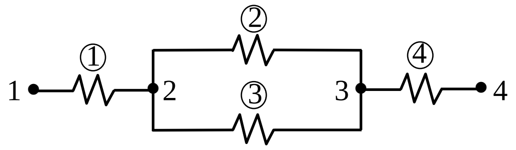
SYSTEM STIFFNESS MATRIX

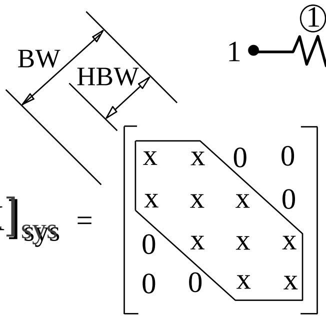
and then combining them to yield the system stiffness matrix by filling values from appropriate rows and columns of the element stiffness matrices,

$$\underset{(4x4)}{[K]_{sys}} = \begin{array}{cc} \begin{array}{cccc} (1) & (2) & (3) & (4) \end{array} & \\ \left[\begin{array}{c|c|c|c} k_1 & -k_1 & & \\ \hline -k_1 & k_1 + k_2 + k_3 & -k_2 - k_3 & \\ \hline & -k_2 - k_3 & k_2 + k_3 + k_4 & -k_4 \\ \hline & & -k_4 & k_4 \end{array}\right] & \begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \end{array} \end{array}$$

Note that this same technique is used in computer programming.

CHARACTERISTICS OF SYSTEM STIFFNESS MATRIX





$$[K]_{sys} = \begin{bmatrix} x & x & 0 & 0 \\ x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix}$$

X = Nonzero term

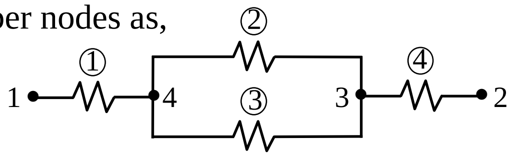
BW = Bandwidth

HBW = Half-bandwidth

- Symmetric, $K_{ij} = K_{ji}$
- Banded with HBW = 2

SYSTEM STIFFNESS BANDWIDTH

If we renumber nodes as,



$$[K]_{sys} = \begin{bmatrix} x & 0 & 0 & x \\ 0 & x & x & 0 \\ 0 & x & x & x \\ x & 0 & x & x \end{bmatrix}$$

Now HBW = 4

- Need more computer memory to store
- Require more computational time to solve
- Most production-type programs include bandwidth optimization capability

SYSTEM STIFFNESS BANDWIDTH

Half-bandwidth can be precomputed prior to FE computation (for general problems) from,

$$\text{HBW} \;=\; (1 + \text{NDIF}) * \text{NDOF}$$

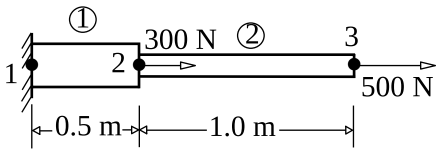
where NDIF is the maximum difference between the node numbers for all elements, and NDOF is the number of degrees of freedom per node.

Here, for first example : $\text{HBW} = (1 + 1) * 1 = 2$

second example : $\text{HBW} = (1 + 3) * 1 = 4$

FINITE ELEMENT EXAMPLE

Bars with different materials and cross-sectional areas subjected to external forces.



Element ① :

$A_1 = 20 \quad \text{cm}^2$

$E_1 = 5\text{x}10^7 \quad \text{kN/m}^2$

Element ② :

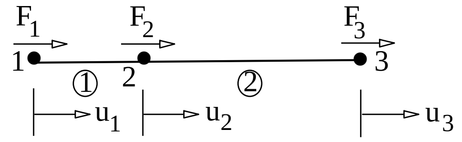
$A_2 = 10 \quad \text{cm}^2$

$E_2 = 10\text{x}10^7 \quad \text{kN/m}^2$

Want
- (a) Nodal displacements at nodes 2 & 3
- (b) Element stresses
- (c) Element forces

FINITE ELEMENT EXAMPLE

The problem is 1-D problem which can be shown as,



where u_1, u_2, u_3 are the nodal displacements and F_1, F_2 , F_3 are the forces that could represent the external or reaction forces. Since the element stiffness matrix for typical element is,

$$[K]_e = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

FINITE ELEMENT EXAMPLE

Thus, the element matrices for element ① & ② are,

$$[K]_{element①} = \begin{array}{c c} \begin{array}{cc} (1) & (2) \end{array} & \\ \begin{bmatrix} \frac{A_1E_1}{L_1} & -\frac{A_1E_1}{L_1} \\ -\frac{A_1E_1}{L_1} & \frac{A_1E_1}{L_1} \end{bmatrix} & \begin{array}{c} (1) \\ (2) \end{array} \end{array} \quad [K]_{element②} = \begin{array}{c c} \begin{array}{cc} (2) & (3) \end{array} & \\ \begin{bmatrix} \frac{A_2E_2}{L_2} & -\frac{A_2E_2}{L_2} \\ -\frac{A_2E_2}{L_2} & \frac{A_2E_2}{L_2} \end{bmatrix} & \begin{array}{c} (2) \\ (3) \end{array} \end{array}$$

which could be combined to yield the system eqs.,

$$\left[\begin{array}{c:c:c} \frac{A_1E_1}{L_1} & \frac{A_1E_1}{L_1} & \\ \hdashline -\frac{A_1E_1}{L_1} & \frac{A_1E_1}{L_1} + \frac{A_2E_2}{L_2} & -\frac{A_2E_2}{L_2} \\ \hdashline & -\frac{A_2E_2}{L_2} & \frac{A_2E_2}{L_2} \end{array}\right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

FINITE ELEMENT EXAMPLE

Substituting values for element ①:

$$\frac{A_1E_1}{L_1} = \frac{(.002)\left(5x10^7\right)}{0.5} = 2x10^5 \quad \text{N/m}$$

and for element ②:

$$\frac{A_2E_2}{L_2} = \frac{(.001)\left(10x10^7\right)}{1.0} = 1x10^5 \quad \text{N/m}$$

leading to the system eqs.,

$$10^5\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

FINITE ELEMENT EXAMPLE

Then, can apply boundary conditions,

Node	Displacement	Force
1	$u_1 = 0$	$F_1 = ?$
2	$u_2 = ?$	$F_2 = 300$
3	$u_3 = ?$	$F_3 = 500$

so that the system eqs. become,

$$10^5\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 300 \\ 500 \end{Bmatrix}$$

Solve to get, $u_2 = 0.004$ m and $u_3 = 0.009$ m

and the reaction $F_1 = -800$ N

FINITE ELEMENT EXAMPLE

Element stresses can then be computed,

$$\sigma_{\text{element①}} = E_1\varepsilon_1 = E_1\frac{u_2 - u_1}{L_1} = \left(5x10^7\right)\frac{(0.004 - 0)}{0.5}$$

$$= 400{,}000 \quad \text{N/m}^2$$

$$\sigma_{\text{element②}} = E_2\varepsilon_2 = E_2\frac{u_3 - u_2}{L_2} = \left(10x10^7\right)\frac{(0.009 - .004)}{1.0}$$

$$= 500{,}000 \quad \text{N/m}^2$$

and also the element forces,

$$\text{Force}_{\text{element①}} = \sigma_1 A_1 = (400{,}000)(.002) = 800 \ \text{N}$$

$$\text{Force}_{\text{element②}} = \sigma_2 A_2 = (500{,}000)(.001) = 500 \ \text{N}$$

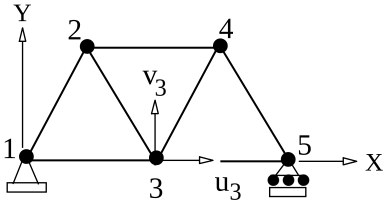
TRANSFORMATION OF FE MATRICES

Problem	Unknown	Type of Unknown	Transformation Required ?
Structural	Displacements	Vector	Yes
Thermal	Temperature	Scalar	No
Fluid	Pressure	Scalar	No
	Velocities	Vector	Yes
Magnetic	Voltage	Scalar	No

Etc.

TRANSFORMATION OF FE MATRICES

Consider a 2-D truss problem,



X-Y = Global coordinates

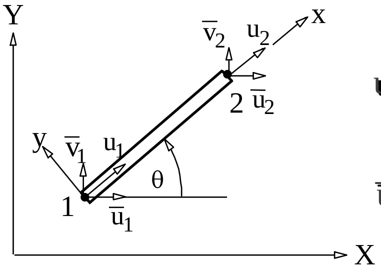
- There are 7 elements and 5 nodes
- Each node has 2 displacement unknowns, u_i & v_i, e.g., u_3 & v_3 at node 3.
- Thus there are 10 unknowns (or 10 eqs) before BC'S
- Total number of equations (NEQ) computed from,

$$\text{NEQ} \quad = \quad \text{NUMNP* NDOF}$$

where NUMNP is total nodes and NDOF is degrees of freedom.

FE MATRIX TRANSFORMATION

Consider a typical element



x-y = Element local coordinates

u_1 & u_2 = Nodal disp in x-direction

X-Y = Global coordinates

$\bar{u}_1$ & $\bar{v}_1$ = Nodal disp components of node 1 in global coord.

From figure,

$$u_1 = \bar{u}_1 \cos\theta + \bar{v}_1 \sin\theta$$
$$u_2 = \bar{u}_2 \cos\theta + \bar{v}_2 \sin\theta$$

Or, in matrix form,

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \end{Bmatrix}$$

FE MATRIX TRANSFORMATION

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \end{Bmatrix}$$

Or, in short, $\underset{(2x1)}{\{u\}} = \underset{(2x4)}{[R]}\ \underset{(4x1)}{\{\bar{u}\}}$

where $\underset{(2x1)}{\{u\}}$ = Vector of nodal displacements in local x-coordinate

$\underset{(4x1)}{\{\bar{u}\}}$ = Vector of nodal displacement components in global X-Y coordinates

$\underset{(2x4)}{[R]}$ = Transformation (or Rotation) matrix

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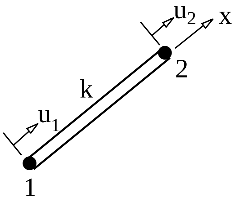
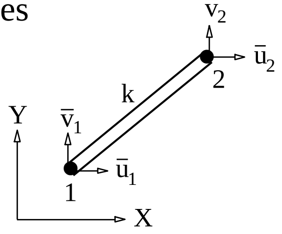
FE MATRIX TRANSFORMATION

Element stiffness matrix in local x-coordinate has dimension of 2x2:



$$\underset{(2x2)}{[K]}\{u\} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Thus the matrix in global X-Y coordinates must have dimension of 4x4:

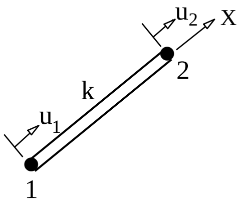


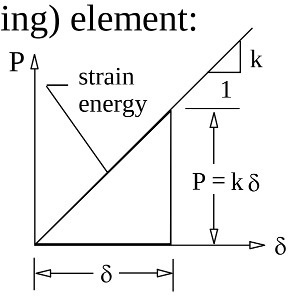
$$\underset{(4x4)}{[\bar{K}]}\{\bar{u}\} = \begin{bmatrix} & & \\ & (4x4) & \\ & & \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \end{Bmatrix}$$

How to derive the element stiffness matrix $\underset{(4x4)}{[\bar{K}]}$?

FE MATRIX TRANSFORMATION

Strain energy for truss (spring) element:





Strain energy $U = \frac{1}{2} k\, \delta^2$ = Area under curve

$$= \frac{1}{2} k \left(u_1 - u_2\right)^2 = \frac{1}{2} k \left(u_1^2 - 2u_1u_2 + u_2^2\right)$$

$$= \frac{1}{2} \underset{(1x2)}{\lfloor u_1 \quad u_2 \rfloor} \underset{(2x2)}{\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}} \underset{(2x1)}{\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}$$

ELEMENT STIFFNESS MATRIX

Or, $U = \frac{1}{2} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T \underbrace{\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}}_{\substack{\text{Stiffness matrix} \\ \text{in local coord.}}} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{2} \underset{(1x2)}{\{u\}^T} \underset{(2x2)}{[K]} \underset{(2x1)}{\{u\}}$

But, from the relation, $\underset{(2x1)}{\{u\}} = \underset{(2x4)}{[R]} \underset{(4x1)}{\{\bar{u}\}}$

Sub., $U = \frac{1}{2} \left([R]\{\bar{u}\}\right)^T [K] \left([R]\{\bar{u}\}\right)$

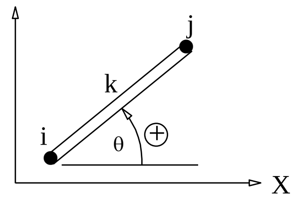
$$U = \frac{1}{2} \underset{(1x4)}{\{\bar{u}\}^T} \underbrace{\underset{(4x2)}{[R]^T} \underset{(2x2)}{[K]} \underset{(2x4)}{[R]}}_{\substack{\text{Stiffness matrix} \\ \text{in global coord.}}} \underset{(4x1)}{\{\bar{u}\}} = \frac{1}{2} \underset{(1x4)}{\{\bar{u}\}^T} \underset{(4x4)}{[\bar{K}]} \underset{(4x1)}{\{\bar{u}\}}$$

Thus, element stiffness matrix in global X-Y coordinates is,

ELEMENT STIFFNESS MATRIX

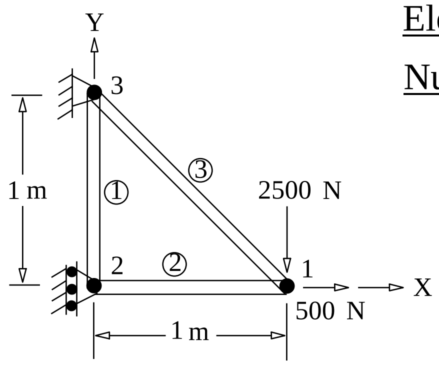
$$\underset{(4x4)}{[\overline{K}]} = \underset{(4x2)}{[R]^T}\underset{(2x2)}{[K]}\underset{(2x4)}{[R]} = \begin{bmatrix}\cos\theta & 0\\ \sin\theta & 0\\ 0 & \cos\theta\\ 0 & \sin\theta\end{bmatrix}\begin{bmatrix}k & -k\\ -k & k\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta & 0 & 0\\ 0 & 0 & \cos\theta & \sin\theta\end{bmatrix}$$

$$[\overline{K}] = k\begin{bmatrix}\cos^2\theta & \sin\theta\cos\theta & -\cos^2\theta & -\sin\theta\cos\theta\\ & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta\\ & & \cos^2\theta & \sin\theta\cos\theta\\ \text{Sym} & & & \sin^2\theta\end{bmatrix}$$



The angle θ is measured from global X-axis (Positive in counterclockwise direction).

TWO-DIMENSIONAL TRUSS EXAMPLE

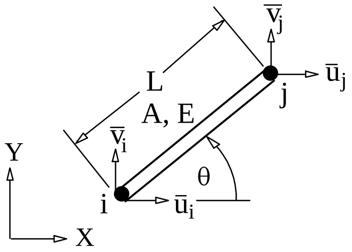


Element Number	Area A (cm^2)	Modulus E (N/cm^2)
1	5.0	$10x10^6$
2	6.0	$30x10^6$
3	4.0	$30x10^6$

Compute nodal displacements and element stresses.

TWO-DIMENSIONAL TRUSS EXAMPLE

First, need to compute $\left[\overline{K}\right]$ for all elements,



$$\underset{(4x4)}{\left[\overline{K}\right]} = \frac{AE}{L}\begin{array}{c} \begin{array}{cccc} (\bar{u}_i) & (\bar{v}_i) & (\bar{u}_j) & (\bar{v}_j) \end{array} \\ \left[\begin{array}{cccc} \cos^2\theta & \sin\theta\cos\theta & -\cos^2\theta & -\sin\theta\cos\theta \\ & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ & & \cos^2\theta & \sin\theta\cos\theta \\ \text{Sym} & & & \sin^2\theta \end{array}\right] \end{array} \begin{array}{c} \\ (\bar{u}_i) \\ (\bar{v}_i) \\ (\bar{u}_j) \\ (\bar{v}_j) \end{array}$$

TWO-DIMENSIONAL TRUSS EXAMPLE

Element	AE/L	i	j	θ	cos θ	sin θ
1	$\frac{(5)(10x10^6)}{100}$	3	2	-90°	0	-1
2	$\frac{(6)(30x10^6)}{100}$	2	1	0°	1	0
3	$\frac{(4)(30x10^6)}{100\sqrt{2}}$	1	3	135°	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$

TWO-DIMENSIONAL TRUSS EXAMPLE

The element stiffness matrices are,

$$[\bar{K}]_{\text{element}\,\textcircled{1}} = \frac{(5)(10\text{x}10^6)}{100} \begin{array}{c} \begin{array}{cccc} (\bar{u}_3) & (\bar{v}_3) & (\bar{u}_2) & (\bar{v}_2) \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ (\bar{u}_3) \\ (\bar{v}_3) \\ (\bar{u}_2) \\ (\bar{v}_2) \end{array}$$

$$[\bar{K}]_{\text{element}\,\textcircled{2}} = \frac{(6)(30\text{x}10^6)}{100} \begin{array}{c} \begin{array}{cccc} (\bar{u}_2) & (\bar{v}_2) & (\bar{u}_1) & (\bar{v}_1) \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ (\bar{u}_2) \\ (\bar{v}_2) \\ (\bar{u}_1) \\ (\bar{v}_1) \end{array}$$

TWO-DIMENSIONAL TRUSS EXAMPLE

$$[\bar{K}]_{\text{element}\,\textcircled{3}} = \frac{(4)(30\text{x}10^6)}{100\sqrt{2}} \begin{array}{c} \begin{array}{cccc} (\bar{u}_1) & (\bar{v}_1) & (\bar{u}_3) & (\bar{v}_3) \end{array} \\ \left[\begin{array}{cccc} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \end{array} \begin{array}{c} \\ (\bar{u}_1) \\ (\bar{v}_1) \\ (\bar{u}_3) \\ (\bar{v}_3) \end{array}$$

Assemble these 3 element matrices leading to system stiffness matrix.

TWO-DIMENSIONAL TRUSS EXAMPLE

The system stiffness matrix is,

$$\underset{(6x6)}{\left[\bar{K}\right]_{sys}} = 10^5 \begin{array}{cccccc|c} (\bar{u}_1) & (\bar{v}_1) & (\bar{u}_2) & (\bar{v}_2) & (\bar{u}_3) & (\bar{v}_3) & \\ \hline 18+3\sqrt{2} & -3\sqrt{2} & -18 & & -3\sqrt{2} & 3\sqrt{2} & (\bar{u}_1) \\ -3\sqrt{2} & 3\sqrt{2} & & & 3\sqrt{2} & -3\sqrt{2} & (\bar{v}_1) \\ -18 & & 18 & & & & (\bar{u}_2) \\ & & & 5 & & -5 & (\bar{v}_2) \\ -3\sqrt{2} & 3\sqrt{2} & & & 3\sqrt{2} & -3\sqrt{2} & (\bar{u}_3) \\ 3\sqrt{2} & -3\sqrt{2} & & -5 & -3\sqrt{2} & 5+3\sqrt{2} & (\bar{v}_3) \end{array}$$

TWO-DIMENSIONAL TRUSS EXAMPLE

Then apply boundary conditions,

Node	Displacement	Force
1	$\bar{u}_1 = ?$	$F_{\bar{u}_1} = 500$
	$\bar{v}_1 = ?$	$F_{\bar{v}_1} = -2,500$
2	$\bar{u}_2 = 0$	$F_{\bar{u}_2} = ?$
	$\bar{v}_2 = ?$	$F_{\bar{v}_2} = 0$
3	$\bar{u}_3 = 0$	$F_{\bar{u}_3} = ?$
	$\bar{v}_3 = 0$	$F_{\bar{v}_3} = ?$

Thus, the system eqs. become,

$$\left[\bar{K}\right]_{sys} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ 0 \\ \bar{v}_2 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 500 \\ -2,500 \\ F_{\bar{u}_2} \\ 0 \\ F_{\bar{u}_3} \\ F_{\bar{v}_3} \end{Bmatrix}$$

TWO-DIMENSIONAL TRUSS EXAMPLE

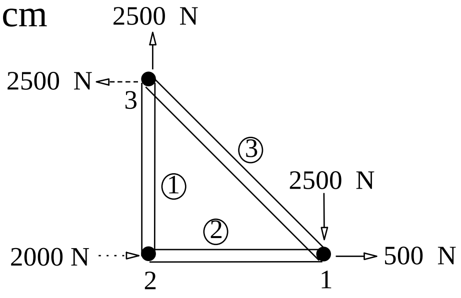
which yield the results of nodal displacements,

$$\bar{u}_1 = -1.11111x10^{-3} \text{ cm}$$

$$\bar{v}_1 = -7.00367x10^{-3} \text{ cm}$$

$$\bar{v}_2 = 0 \text{ cm}$$

and reaction forces,



$$F_{\bar{u}_2} = 2000 \text{ N}$$

$$F_{\bar{u}_3} = -2500 \text{ N}$$

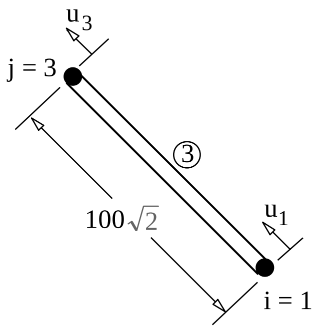
$$F_{\bar{v}_3} = 2500 \text{ N}$$

that can be checked for equilibrium.

TWO-DIMENSIONAL TRUSS EXAMPLE

Element stresses are computed from,

$$\sigma = E\,\varepsilon = E\,\frac{u_j - u_i}{L}$$



As an example, the stress in element ③ :

$$\sigma_{element③} = \left(30x10^6\right)\frac{\left(u_3 - u_1\right)}{100\sqrt{2}}$$

Here, u_1 and u_3 are nodal displacements in the element local coordinate which can be determined from the relation $\underset{(2x1)}{\{u\}} = \underset{(2x4)}{[R]}\,\underset{(4x1)}{\{\bar{u}\}}$

TWO-DIMENSIONAL TRUSS EXAMPLE

i,e.,

$$\begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_3 \\ \bar{v}_3 \end{Bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{Bmatrix} -1.1111x10^{-3} \\ -7.1111x10^{-3} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -4.16667x10^{-3} \\ 0 \end{Bmatrix}$$

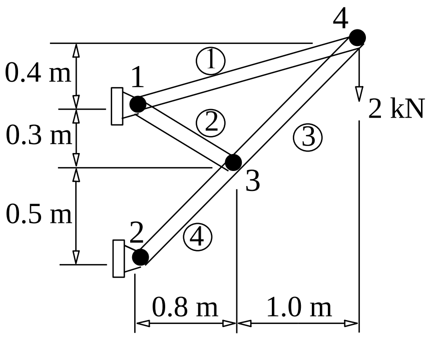
Thus, the stress in element ③ is

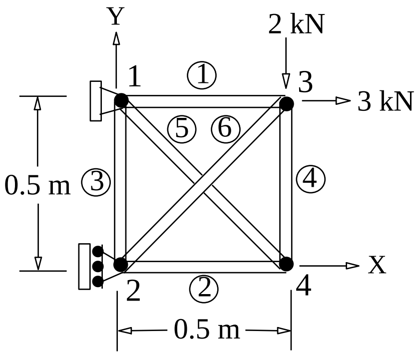
$$\sigma_{element ③} = \left(30x10^6\right)\frac{\left(u_3 - u_1\right)}{100\sqrt{2}} = \left(30x10^6\right)\frac{0 + 4.16667x10^{-3}}{100\sqrt{2}}$$

$$= 884 \quad N/cm^2$$

TWO-DIMENSIONAL TRUSS PROBLEMS

The same procedure described in this example can be applied to solve other problems with more complex configuration:





CONCLUSION OF DIRECT APPROACH

Advantages:

- Element equations are easy to derive and understand
- Applicable to other 1-D problems (e.g., heat transfer, flow in pipe, etc.)

Limitation:

- Can not extend to 2 & 3 dimensional continuum problems
- Need other approaches to derive FE equations