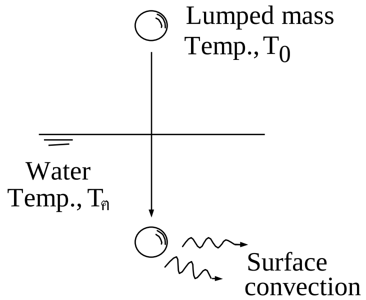
METHOD OF WEIGHTED RESIDUALS

METHOD OF WEIGHTED RESIDUALS

To understand the Method of Weighted Residuals (MWR), let's consider the example of transient thermal response of a lumped mass. Given,



A = surface area
V = volume
ρ = density
c = specific heat
h = convection coefficient
τ = time
T = Temperature

Want: Lumped mass temperature, $T = T(\tau) = ?$

TRANSIENT RESPONSE OF LUMPED MASS

Conservation of energy:

Energy in - Energy out = Energy stored

$$0 - hA(T - T_\infty) = \rho c V \frac{\partial T}{\partial \tau}$$

$$\rho c V \frac{\partial T}{\partial \tau} + hA(T - T_\infty) = 0$$

with initial condition: $T(\tau = 0) = T_0$

To easily solve this ODE, let's denote,

Non-dimensional temperature $\bar{x} = \dfrac{T - T_\infty}{T_0 - T_\infty}$

Non-dimensional time $t = \dfrac{hA\tau}{\rho c V}$

then the ODE & IC become,

TRANSIENT RESPONSE OF LUMPED MASS

ODE: $\dfrac{d\bar{x}}{dt} + \bar{x} = 0$

IC: $\bar{x}(t = 0) = 1$

To solve for exact solution, $\bar{x}$(t), we separate the variables,

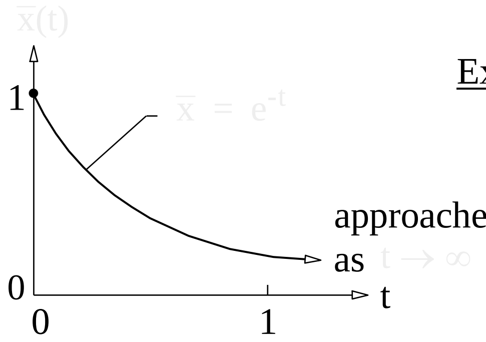
$$\frac{d\bar{x}}{\bar{x}} = -dt$$

then integrate $\ln \bar{x} = -t + B$

where B is the integrating constant which is zero after applying the initial condition. Thus, the exact solution is,

$$\bar{x} = \bar{x}(t) = e^{-t}$$

TRANSIENT RESPONSE OF LUMPED MASS



Exact solution:

$$\bar{x} = \bar{x}(t) = e^{-t}$$

Now, if we don't know the exact solution, and we want to use the method of weighted residuals to find approximate solution for $0 < t < 1$.

METHOD OF WEIGHTED RESIDUALS

Assume an approximate solution in the form,

$$x(t) = 1 + C_1 t + C_2 t^2$$

where C_1 and C_2 are constants to be determined. Note that if we substitute this approx. sol. into the diff. eq.

$$\frac{dx}{dt} + x \neq 0 \qquad \text{in fact} \qquad = R$$

i.e., $R = (0 + C_1 + 2C_2 t) + (1 + C_1 t + C_2 t^2)$

or, $R = R(t) = 1 + C_1(1 + t) + C_2(2t + t^2)$

So, the idea is to determine C_1 and C_2 such that the Residual $R \to 0$.

METHOD OF WEIGHTED RESIDUALS

Four approaches:

(1) Point collocation Set residuals at selected points to be zero.

(2) Subdomain collocation Set areas under the residual curve to be zero.

(3) Galerkin Include weighting functions. Widely used in FE method.

(4) Least squares Square the residual with minimization process.

POINT COLLOCATION

Set the residuals at selected points to be zero, i.e.,

$$R(t_i) = 0 \qquad i = 1, 2, ..., \text{no. of unknowns}$$

Since there are two unknowns of C_1 and C_2, thus we need 2 eqs. of, $R(t_1) = 0$ and $R(t_2) = 0$

where t_1 and t_2 should be in the range of consider 0 to 1. E.g., select,

$$t_1 = \frac{1}{3}; \quad R\left(\frac{1}{3}\right) = 1 + C_1\left(1 + \frac{1}{3}\right) + C_2\left(2\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2\right) = 0$$

$$t_2 = \frac{2}{3}; \quad R\left(\frac{2}{3}\right) = 1 + C_1\left(1 + \frac{2}{3}\right) + C_2\left(2\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2\right) = 0$$

Solve to get, $C_1 = -0.9310$ and $C_2 = 0.3103$

Then, $x(t) = 1 - 0.9310\, t + 0.3103\, t^2$

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SUBDOMAIN COLLOCATION

Set areas under the residual curve to be zero. Since there are 2 unknowns, thus we need 2 eqs., e.g.

$$\int_0^{t_1} R(t)\,dt = 0 \quad \text{and} \quad \int_{t_1}^{1} R(t)\,dt = 0$$

If we select $t_1 = \frac{1}{2}$, then,

$$\int_0^{1/2} R(t)\,dt\,; \quad \frac{1}{2} + \frac{5}{8}C_1 + \frac{7}{24}C_2 = 0$$

$$\int_{1/2}^{1} R(t)\,dt\,; \quad \frac{1}{2} + \frac{7}{8}C_1 + \frac{25}{24}C_2 = 0$$

Solve to get, $C_1 = -0.9474$ and $C_2 = 0.3158$

Then, the approximate solution is,

$$x(t) = 1 - 0.9474\,t + 0.3158\,t^2$$

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GALERKIN APPROACH

Multiply the residual by some weighting functions, perform integration over the entire domain, and set to be zero.

$$\int_0^1 R(t)\,W_i(t)\,dt = 0 \qquad i = 1.\ 2, \ldots, \text{No. of unknowns}$$

where $W_i(t)$ is the weighting functions which are normally selected as the terms that weight with the unknowns, i.e.,

$$x(t) = 1 + C_1\underbrace{t}_{W_1} + C_2\underbrace{t^2}_{W_2}$$

here, we use $W_i = t$ and $W_2 = t^2$, then substitute into the above eq. to get,

GALERKIN APPROACH

$$\int_0^1 R(t)\, t\, dt = 0 ; \quad \frac{1}{2} + \frac{5}{6}C_1 + \frac{11}{12}C_2 = 0$$

$$\int_0^1 R(t)\, t^2\, dt = 0 ; \quad \frac{1}{3} + \frac{7}{12}C_1 + \frac{7}{10}C_2 = 0$$

Solve to get, $C_1 = -0.9143$ and $C_2 = 0.2857$

Thus, the approximate solution is,

$$x(t) = 1 - 0.9143\, t + 0.2857\, t^2$$

Note: We will use this approach to derive FE eqs. later. Since distribution of element variable is,

$$\phi = \underbrace{N_1}_{W_1} \phi_1 + \underbrace{N_2}_{W_2} \phi_2$$

Here, the nodal unknowns ϕ_1 and ϕ_2 are similar to C_1 and C_2. The element interpolation functions N_1 and N_2 will be selected as the weighting functions.

LEAST SQUARES APPROACH

Square the residual, perform integration over entire domain, and minimize with respect to the unknowns, i.e.,

$$\frac{\partial}{\partial C_i} \int_0^1 R^2(t)\, dt = 0 \qquad i = 1.\ 2, ..., \text{No. of unknowns}$$

Since there are 2 unknowns, thus we have 2 eqs.,

$$\frac{\partial}{\partial C_1} \int_0^1 R^2(t)\, dt = 0 ; \quad \frac{3}{2} + \frac{7}{3}C_1 + \frac{9}{4}C_2 = 0$$

$$\frac{\partial}{\partial C_2} \int_0^1 R^2(t)\, dt = 0 ; \quad \frac{4}{3} + \frac{9}{4}C_1 + \frac{38}{15}C_2 = 0$$

Solve to get, $C_1 = -0.9427$ and $C_2 = 0.3110$

Then, the approximate solution is,

$$x(t) = 1 - 0.9427\, t + 0.3110\, t^2$$

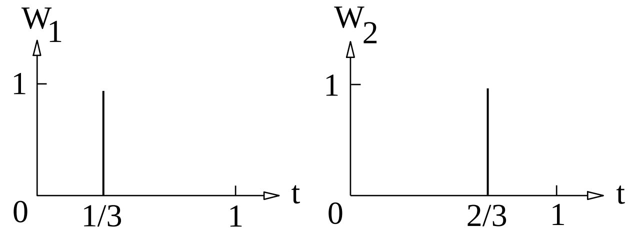
METHOD OF WEIGHTED RESIDUALS

In conclusion, these four approaches can be written in the form,

$$\int_0^1 R(t)\, W_i(t)\, dt = 0$$

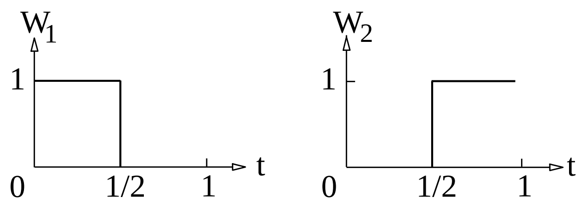
where W_i are the weighting functions that can be described as follows,

"Point collocation"

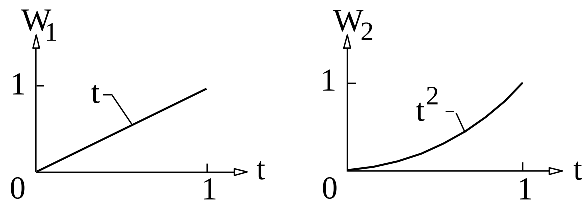


METHOD OF WEIGHTED RESIDUALS

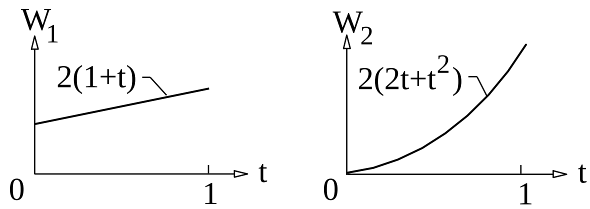
"Subdomain collocation"



"Galerkin"

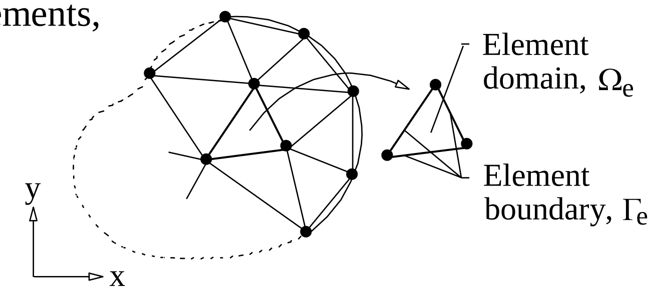


"Least squares"



GENERAL PROCEDURE OF MWR

Step 1: Discretize the given domain into a number of elements,



Then identify the governing differential eq(s) of the problem,

$$L(\bar{\phi}) = 0$$

where L denotes the differential operator and $\bar{\phi}$ is the exact solution.

GENERAL PROCEDURE OF MWR

Step 2: Assume element approximate solution in the form,

$$\phi = \phi(x,y) = \sum_{i=1}^{m} N_i \phi_i = \underset{(1xm)}{\lfloor N \rfloor} \underset{(1xm)}{\{\phi\}}$$

where $\phi = \phi(x,y)$ = element approximate solution
$N_i = N_i(x,y)$ = element interpolation functions
ϕ_i = element nodal unknowns
m = number of element nodes

Step 3: Apply the method of weighted residuals for a typical element. If we substitute the approximate solution, $\phi(x,y)$, into the governing diff. eq.,

$$L(\phi) \neq 0 \quad \text{in general, but} \quad L(\phi) = \text{Error} = R$$

GENERAL PROCEDURE OF MWR

Here R is called the Residual, i.e.,

$$R \;=\; L(\phi) \;=\; L\left(\lfloor N \rfloor \{\phi\}\right) \;=\; L\left(\sum_{i=1}^{m} N_i\,\phi_i\right)$$

Then apply the method of weighted residuals (Gakerkin approach):

Multiply the Residual R by some weighting functions, perform integration over element domain, and set to be zero.

$$\int_{\Omega^{(e)}} W_i\, R\, d\Omega \;=\; 0 \qquad i = 1, 2, ..., m$$

Note : If $W_i = N_i$ ⇨ called “Bubnov-Galerkin” approach
$W_i \neq N_i$ ⇨ called “Petrov-Galerkin” approach

GENERAL PROCEDURE OF MWR

Step 4: Perform integration by parts (or use Gauss’s theorem).

$$\int_{\Omega^{(e)}} W_i\, R\, d\Omega \;=\; \int_{\Omega^{(e)}} W_i\, L\left(\sum_{i=1}^{m} N_i\,\phi_i\right) d\Omega$$

$$= \underbrace{\int_{\Omega^{(e)}} \left(W_i, N_i, \phi_i\right) d\Omega}_{\substack{\text{Associated with} \\ \text{element domain, } \Omega^{(e)}}} \;+\; \underbrace{\int_{\Gamma^{(e)}} \left(W_i, N_i, \phi_i\right) d\Gamma}_{\substack{\text{Associated with} \\ \text{element boundary, } \Gamma^{(e)}}} \;=\; 0$$

Step 5: Apply boundary conditions on the element boundary integral term $\int_{\Gamma^{(e)}}$ as needed.

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GENERAL PROCEDURE OF MWR

Step 6: Write the element eqs. in matrix form,

$$[K]\{\phi\} = \{F\}$$

(mxm) (mx1) (mx1)

where $[K]$ = Element stiffness matrix

$\{\phi\}$ = Vector of element nodal unknowns

$\{F\}$ = Vector of element nodal loads

Then assemble all element eqs., apply all BC's on the system eqs., and solve for nodal unknowns.

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METHOD OF WEIGHTED RESIDUALS

Objective: Derive finite element equations from governing differential equation.

Simplest example is 1-D Poisson's eq.:

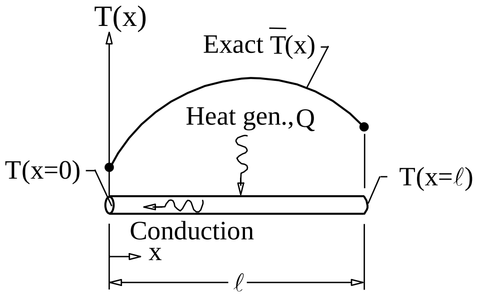
$$\frac{d^2u}{dx^2} = f(x)$$

where u = u(x). Such differential eq. may represent the problems of,

- Bar with its own weight
- 1-D viscous flow
- 1-D heat conduction with internal heat generation

etc.

CONDUCTION WITH HEAT GENERATION



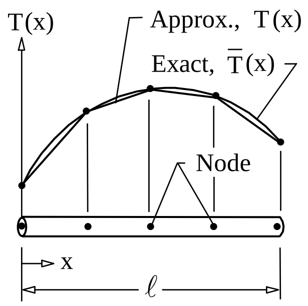
Differential equation:

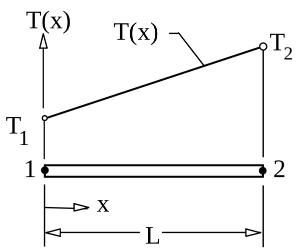
$$kA \frac{d^2\bar{T}}{dx^2} = -QA$$

where k is thermal conductivity and A is cross-sectional area.

Boundary conditions: $T(x = 0) = T_0$ and $T(x = \ell) = T_\ell$

DISCRETIZATION & INTERPOLATION FUNC.





Assume linear element temperature distribution,

$$T(x) = ax + b$$

At $x = 0$; $T(x = 0) = T_1 = b$

At $x = L$; $T(x = L) = T_2 = aL + b$

Then $a = \dfrac{T_2 - T_1}{L}$ and $b = T_2$

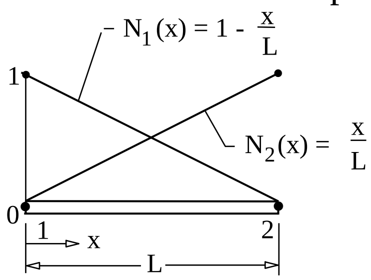
ELEMENT INTERPOATION FUNCTIONS

Then $T(x) = \left(1 - \frac{x}{L}\right) T_1 + \left(\frac{x}{L}\right) T_2$

which can be written in the form,

$$T(x) = N_1 T_1 + N_2 T_2 = \lfloor N_1 \quad N_2 \rfloor \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \underset{(1x2)}{\lfloor N \rfloor} \; \underset{(2x1)}{\{T\}}$$

where the element interpolation functions,



$$N_1 = 1 - \frac{x}{L}$$

$$N_2 = \frac{x}{L}$$

with properties of,

$$N_i = \begin{cases} 1 & \text{at node i} \\ 0 & \text{at other node} \end{cases}$$

METHOD OF WEIGHTED RESIDUALS

If we can solve for exact solution, $\overline{T}$, and substitute into the differential eq., then RHS will be identically zero, i.e.,

$$kA \frac{d^2\overline{T}}{dx^2} + QA = 0$$

However, in general, we don't know exact solution. Thus, if we substitute the approximate solution, T, RHS will not be zero, but will be equal to a Residual, R:

$$kA \frac{d^2T}{dx^2} + QA = R$$

The idea is thus to minimize R so that error is minimum.

METHOD OF WEIGHTED RESIDUALS

The method of weighted residuals is to multiply R by some weighting functions W_i, *perform integration over element domain, and set to zero,* i.e.:

$$\int_0^L W_i \, R \, dx = 0$$

Since the element has two unknowns, thus we need 2 eqs., i.e., i = 1, 2. Substitute the residual to get,

$$\int_0^L W_i \left(kA \frac{d^2T}{dx^2} + QA \right) dx = 0$$

Expand,

$$\int_0^L W_i \, k \, A \frac{d^2T}{dx^2} dx + \int_0^L W_i \, Q \, A \, dx = 0 \qquad i = 1, 2$$

METHOD OF WEIGHTED RESIDUALS

Apply integration by parts to the first integral term in order to produce the boundary term, i.e.,

$$\int_0^L \underbrace{W_i}_{u} \, \underbrace{k \, A \frac{d^2T}{dx^2} dx}_{dv} = W_i \, k \, A \frac{dT}{dx}\Bigg|_0^L - \int_0^L k \, A \frac{dT}{dx} \frac{dW_i}{dx} dx$$

by using the formula, $\int_0^L u \, dv = u \, v \Big|_0^L - \int_0^L v \, du$

where $u = W_i$ then $du = \frac{dW_i}{dx} dx$

and $dv = k \, A \frac{d^2T}{dx^2} dx$ then $v = k \, A \frac{dT}{dx}$

Then, the element eqs. becomes,

METHOD OF WEIGHTED RESIDUALS

$$\int_0^L kA \frac{dW_i}{dx}\frac{dT}{dx}dx = W_i kA \frac{dT}{dx}\Big|_0^L + \int_0^L W_i Q A\, dx$$

Recall $T = \lfloor N_1 \quad N_2 \rfloor \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$ then $\frac{dT}{dx} = \left\lfloor \frac{dN_1}{dx} \quad \frac{dN_2}{dx} \right\rfloor \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$

Substitute and write out for i = 1, 2 to get 2 equations as,

$$i = 1;\ \int_0^L kA\frac{dW_1}{dx}\left\lfloor \frac{dN_1}{dx} \quad \frac{dN_2}{dx} \right\rfloor \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} dx = W_1 kA \frac{dT}{dx}\Big|_0^L + \int_0^L W_1 Q A\, dx$$

$$i = 2;\ \int_0^L kA\frac{dW_2}{dx}\left\lfloor \frac{dN_1}{dx} \quad \frac{dN_2}{dx} \right\rfloor \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} dx = W_2 kA \frac{dT}{dx}\Big|_0^L + \int_0^L W_2 Q A\, dx$$

METHOD OF WEIGHTED RESIDUALS

For standard FE., we select $W_i = N_i$ which is known as Bubnov-Galerkin approach, we will get element equations in the form,

$$\int_0^L kA \begin{Bmatrix} \frac{dN_1}{dx} \\ \frac{dN_2}{dx} \end{Bmatrix} \left\lfloor \frac{dN_1}{dx} \quad \frac{dN_2}{dx} \right\rfloor dx \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$= \left(\begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} kA \frac{dT}{dx} \right)\Bigg|_0^L + \int_0^L \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} Q A\, dx$$

METHOD OF WEIGHTED RESIDUALS

Or, in short: $\left[K_C\right]\{T\} = \{Q_C\} + \{Q_Q\}$

where $\left[K_C\right]$ = Element conduction matrix

$\{T\}$ = Vector of element nodal temperatures

$\{Q_C\}$ = Vector of element nodal heat fluxes

$\{Q_Q\}$ = Vector of element nodal heat generations

METHOD OF WEIGHTED RESIDUALS

Conduction matrix $\quad \left[K_C\right] = \int_0^L k\,A \begin{Bmatrix} \frac{dN_1}{dx} \\ \frac{dN_2}{dx} \end{Bmatrix} \left\lfloor \frac{dN_1}{dx} \quad \frac{dN_2}{dx} \right\rfloor dx$

Since $\quad N_1 = 1 - \frac{x}{L} \quad$ and $\quad N_2 = \frac{x}{L}$

then $\quad \frac{dN_1}{dx} = -\frac{1}{L} \quad$ and $\quad \frac{dN_2}{dx} = \frac{1}{L}$

Substitute $\left[K_C\right] = \int_0^L k\,A \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} \left\lfloor -\frac{1}{L} \quad \frac{1}{L} \right\rfloor dx = \int_0^L \frac{k\,A}{L^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx$

If thermal conductivity k & cross-sectional area A are constant, then

$$\left[K_C\right] = \frac{k\,A}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

ELEMENT LOAD VECTORS

Load vector due to internal heat generation

$$\{Q_Q\} = \int_0^L \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} Q\,A\,dx = \int_0^L \begin{Bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{Bmatrix} Q\,A\,dx = Q\,A\,L \begin{Bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{Bmatrix}$$

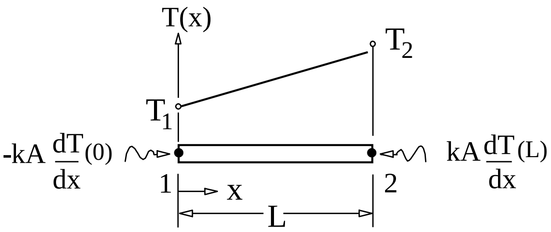
Load vector due to nodal heat fluxes

$$\{Q_C\} = \left(\begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} k\,A \frac{dT}{dx} \right) \Bigg|_0^L = \begin{Bmatrix} \left(N_1\,k\,A \frac{dT}{dx} \right) \Big|_0^L \\ \left(N_2\,k\,A \frac{dT}{dx} \right) \Big|_0^L \end{Bmatrix}$$

$$= \begin{Bmatrix} N_1(L)\,kA \frac{dT}{dx}(L) - N_1(0)\,kA \frac{dT}{dx}(0) \\ N_2(L)\,kA \frac{dT}{dx}(L) - N_2(0)\,kA \frac{dT}{dx}(0) \end{Bmatrix}$$

But $N_1(0) = 1,\ N_1(L) = 0,\ N_2(0) = 0,\ N_2(L) = 1$, thus

ELEMENT LOAD VECTORS



$$\{Q_C\} = \begin{Bmatrix} -k\,A \frac{dT}{dx}(0) \\ k\,A \frac{dT}{dx}(L) \end{Bmatrix}$$

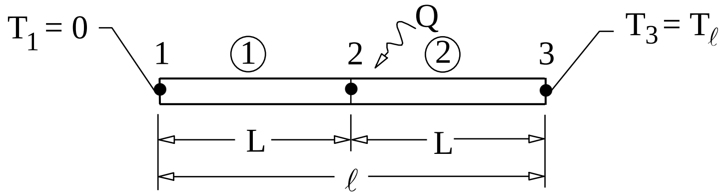
Conclusion of element equations:

$$[K_C]\{T\} = \{Q_C\} + \{Q_Q\}$$

$$\frac{k\,A}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} -k\,A \frac{dT}{dx}(0) \\ k\,A \frac{dT}{dx}(L) \end{Bmatrix} + Q\,A\,L \begin{Bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{Bmatrix}$$

METHOD OF WEIGHTED RESIDUALS

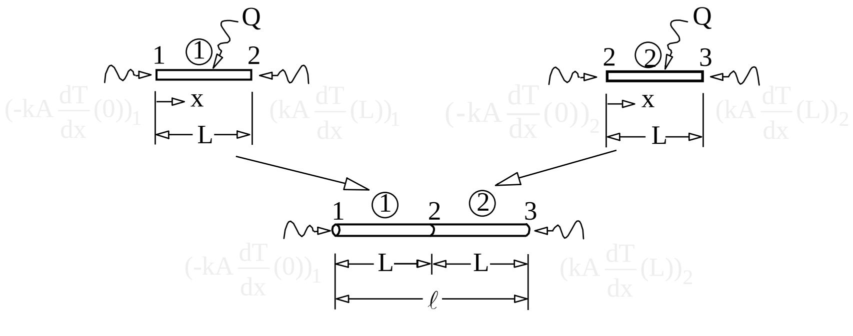
Example   A rod divided into 2 elements with specified temp. at both ends as shown is subjected to internal heat generation Q. Compute the temperature at node ②.



Typical element eqs., e.g., for element no. ①:

$$\frac{kA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = \begin{Bmatrix}(-kA\frac{dT}{dx}(0))_1\\ (kA\frac{dT}{dx}(L))_1\end{Bmatrix} + QAL\begin{Bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{Bmatrix}$$

ASSEMBLING OF ELEMENT EQUATIONS



$$\frac{kA}{L}\begin{bmatrix}1 & -1 & 0\\ -1 & 1+1 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}\left(-kA\frac{dT}{dx}(0)\right)_1\\ \left(kA\frac{dT}{dx}(L)\right)_1 + \left(-kA\frac{dT}{dx}(0)\right)_2\\ \left(kA\frac{dT}{dx}(L)\right)_2\end{Bmatrix} + QAL\begin{Bmatrix}\frac{1}{2}\\ \frac{1}{2}+\frac{1}{2}\\ \frac{1}{2}\end{Bmatrix}$$

SYSTEM OF EQUATIONS

Heat fluxes at node 2 must be continuous, and apply BC's:

$$\frac{kA}{L}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}T_1 = 0\\ T_2 = ?\\ T_3 = T_\ell\end{Bmatrix} = \begin{Bmatrix}(-kA\frac{dT}{dx}(0))_1\\ 0\\ (kA\frac{dT}{dx}(L))_2\end{Bmatrix} + QAL\begin{Bmatrix}\frac{1}{2}\\ 1\\ \frac{1}{2}\end{Bmatrix}$$

Then use second eq. to solve for temperature at node 2,

$$\frac{kA}{L}(0 + 2T_2 - T_\ell) = 0 + QAL$$

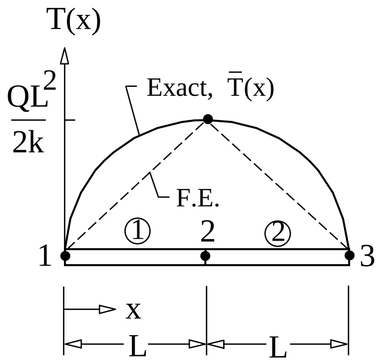
$$T_2 = \frac{QL^2}{2k} + \frac{T_\ell}{2}$$

Note: Identical procedure can be used for more elements.

USE OF SOLUTION SYMMETRY

If $T_\ell = 0$, then the temperature at node 2 becomes,

$$T_2 = \frac{QL^2}{2k}$$



Note that the exact temperature distribution in this case is,

$$\bar{T}(x) = \frac{QL^2}{2k}\left(2\frac{x}{L} - \frac{x^2}{L^2}\right)$$

which is symmetry.

USE OF SOLUTION SYMMETRY

Due to symmetry of solution, only half of the rod can be used in modeling. Here, if use 1 element,

$$\frac{kA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = \begin{Bmatrix}(-kA\frac{dT}{dx}(0))_1\\ (kA\frac{dT}{dx}(L))_1\end{Bmatrix} + QAL\begin{Bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{Bmatrix}$$

and apply the condition of no heat flow across node 2, i.e.,

$$\left(kA\frac{dT}{dx}(L)\right)_1 = 0$$

Then the element eqs. become,

$$\frac{kA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}T_1 = 0\\ T_2 = ?\end{Bmatrix} = \begin{Bmatrix}(-kA\frac{dT}{dx}(0))_1\\ 0\end{Bmatrix} + QAL\begin{Bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{Bmatrix}$$

USE OF SOLUTION SYMMETRY

Then use the second eq. to solve for T_2,

$$\frac{kA}{L}(0 + T_2) = 0 + \frac{QAL}{2}$$

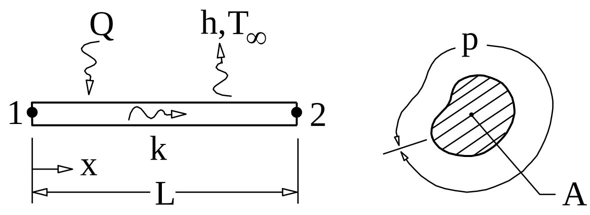
$$T_2 = \frac{QL^2}{2k}$$

<u>Remark</u>: For practical problems in 3-D, always take advantage of solution symmetry in order to reduce total number of elements and unknowns.

Problem	Reduction factor
1-D	2
2-D	4
3-D	8

ROD WITH SURFACE CONVECTION

If the rod has additional surface convection,



The corresponding differential equation is,

$$k\,A\,\frac{d^2T}{dx^2} - h\,p\,(T - T_\infty) + QA = 0$$

where h = convection coefficient

p = perimeter

T_∞ = surrounding medium temperature

ROD WITH SURFACE CONVECTION

Apply MWR,

$$\int_0^L W_i\,R\,dx = 0 \qquad i = 1, 2$$

$$\int_0^L W_i\,(k\,A\,\frac{d^2T}{dx^2} - h\,p\,(T - T_\infty) + Q\,A)\,dx = 0$$

$$\int_0^L W_i\,k\,A\,\frac{d^2T}{dx^2}\,dx + \int_0^L W_i\,Q\,A\,dx$$
$$- \int_0^L W_i\,h\,p\,T\,dx + \int_0^L W_i\,h\,p\,T_\infty\,dx = 0$$

with $W_i = N_i$ and $T = \lfloor N \rfloor \{T\}$, the last two integral terms become,

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ROD WITH SURFACE CONVECTION

$$\int_0^L W_i\, h\, p\, T\, dx \quad \Rightarrow \quad \int_0^L h\, p\, \{N\}\lfloor N \rfloor dx\, \{T\}$$

$$= \lfloor K_h \rfloor \{T\}$$

$$\int_0^L W_i\, h\, p\, T_\infty\, dx \quad \Rightarrow \quad \int_0^L h\, p\, T_\infty\, \{N\} dx$$

$$= \{Q_h\}$$

where $\lfloor K_h \rfloor$ = Convection matrix

$\{Q_h\}$ = Convection load vector

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ROD WITH SURFACE CONVECTION

The finite element eqs. are in the form,

$$\lfloor K_c \rfloor \{T\} + \lfloor K_h \rfloor \{T\} \quad = \quad \{Q_c\} + \{Q_Q\} + \{Q_h\}$$

or, in detail,

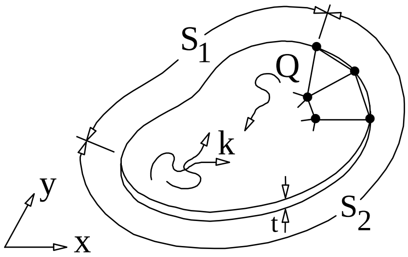
$$\frac{k\,A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} + h\,p\,L\begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} -k\,A\frac{dT}{dx}(0) \\ k\,A\frac{dT}{dx}(L) \end{Bmatrix} + QAL\begin{Bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{Bmatrix} + h\,p\,L\,T_\infty \begin{Bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{Bmatrix}$$

which can be used in many applications, such as heat transfer in motor fins, etc.

TWO-DIMENSIONAL HEAT TRANSFER

Plate with Internal Heat Generation



Governing differential equation:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) = -Q$$

Boundary conditions:

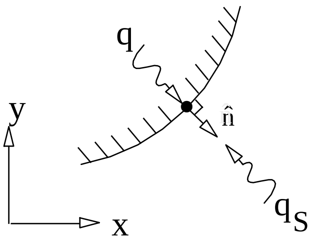
(1) Specified temperature along edge S_1:

$$T(x, y) = T_1(x, y)$$

TWO-DIMENSIONAL HEAT TRANSFER

(2) Specified surface heating along edge S_2. From Fourier's law,

$$q = -k\frac{\partial T}{\partial x}n_x - k\frac{\partial T}{\partial y}n_y$$



where n_x and n_y are direction cosines of unit vector $\hat{n}$ normal to the edge. For a specified heating q_s into edge, then

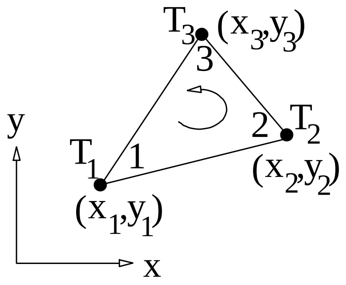
$$q_s = -q = k\frac{\partial T}{\partial x}n_x + k\frac{\partial T}{\partial y}n_y$$

This means, for insulated edge,

$$k\frac{\partial T}{\partial x}n_x + k\frac{\partial T}{\partial y}n_y = 0$$

TRIANGULAR ELEMENT

Element Interpolation Functions



Assume a flat temperature distribution over the element:

$$T(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y$$

where α_i, i = 1, 2, 3, are constants to be determined from conditions at nodes,

At node 1: $T(x_1, y_1) = T_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$

At node 2: $T(x_2, y_2) = T_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$

At node 3: $T(x_3, y_3) = T_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$

Solve for α_i, i = 1, 2, 3, and rearrange terms to get,

TRIANGULAR ELEMENT

$$T(x, y) = \lfloor N_1(x, y) \; N_2(x, y) \; N_3(x, y) \rfloor \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \underset{(1x3)(3x1)}{\lfloor N \rfloor \{T\}}$$

where $\{T\}$ is the vector of nodal temperatures, and $\lfloor N \rfloor$ is the element interpolation matrix defined by,

$$N_i(x, y) = \frac{1}{2A}(a_i + b_i x + c_i y) \qquad i = 1, 2, 3$$

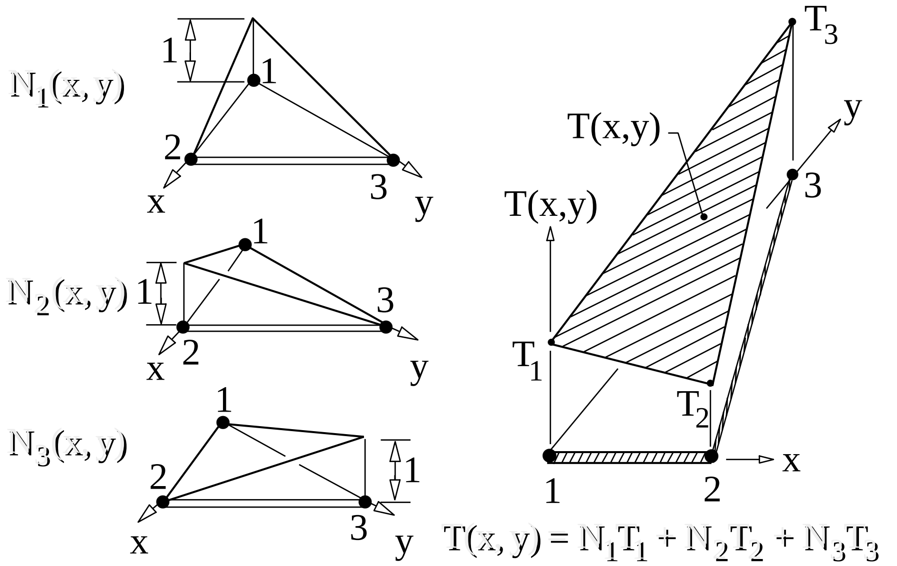
Here A = Area of triangle

$$= \frac{1}{2}\left[x_2(y_3 - y_1) + x_1(y_2 - y_3) + x_3(y_1 - y_2)\right]$$

and

$$a_1 = x_2 y_3 - x_3 y_2 \qquad b_1 = y_2 - y_3 \qquad c_1 = x_3 - x_2$$
$$a_2 = x_3 y_1 - x_1 y_3 \qquad b_2 = y_3 - y_1 \qquad c_2 = x_1 - x_3$$
$$a_3 = x_1 y_2 - x_2 y_1 \qquad b_3 = y_1 - y_2 \qquad c_3 = x_2 - x_1$$

TRIANGULAR ELEMENT INTERPOLATIONS



$$T(x, y) = N_1T_1 + N_2T_2 + N_3T_3$$

DERIVATION OF ELEMENT EQUATIONS

If $\overline{T}$ is the exact temperature solution of the plate, then

$$\frac{\partial}{\partial x}\left(k\frac{\partial \overline{T}}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial \overline{T}}{\partial y}\right) + Q \;=\; 0$$

Since we use approximate T, then RHS of the above equation is not zero, but is equal to a Residual R, i.e.

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + Q \;=\; R$$

Apply MWR:
$$\int_{\Omega^{(e)}} W_i\, R\, d\Omega \;=\; 0$$

where $\Omega^{(e)}$ is the element volume. Substitute,

$$\int_{\Omega^{(e)}} W_i \left(\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + Q\right) d\Omega \;=\; 0$$

DERIVATION OF ELEMENT EQUATIONS

Expand,

$$\int_{\Omega^{(e)}} W_i \left(\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y} \right) \right) d\Omega + \int_{\Omega^{(e)}} W_i\, Q\, d\Omega \;=\; 0$$

As in 1-D, we apply integration by parts to the first term, Here is to use Gauss's theorem,

$$\int_{\Omega^{(e)}} u\,(\nabla \cdot \vec{V})\, d\Omega \;=\; \int_{\Gamma^{(e)}} u\,(\vec{V} \cdot \hat{n})\, d\Gamma \;-\; \int_{\Omega^{(e)}} (\nabla u \cdot \vec{V})\, d\Omega$$

Let $u = W_i$

$$\left.\begin{aligned} \nabla &= \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} \\ \vec{V} &= k\frac{\partial T}{\partial x}\hat{i} + k\frac{\partial T}{\partial y}\hat{j} \end{aligned}\right\} (\nabla \cdot \vec{V}) \;=\; \left(\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y}) \right)$$

DERIVATION OF ELEMENT EQUATIONS

Then the terms on RHS of Gauss's theorem which states as,

$$\int_{\Omega^{(e)}} u\,(\nabla \cdot \vec{V})\, d\Omega \;=\; \int_{\Gamma^{(e)}} u\,(\vec{V} \cdot \hat{n})\, d\Gamma \;-\; \int_{\Omega^{(e)}} (\nabla u \cdot \vec{V})\, d\Omega$$

using $\hat{n} = n_x\hat{i} + n_y\hat{j}$, are:

$$\vec{V} \cdot \hat{n} \;=\; k\frac{\partial T}{\partial x} n_x + k\frac{\partial T}{\partial y} n_y$$

$$u(\vec{V} \cdot \hat{n}) \;=\; W_i\,(k\frac{\partial T}{\partial x} n_x + k\frac{\partial T}{\partial y} n_y)$$

$$\nabla u \;=\; \frac{\partial W_i}{\partial x}\hat{i} + \frac{\partial W_i}{\partial y}\hat{j}$$

$$\nabla u \cdot \vec{V} \;=\; \frac{\partial W_i}{\partial x} k\frac{\partial T}{\partial x} + \frac{\partial W_i}{\partial y} k\frac{\partial T}{\partial y}$$

DERIVATION OF ELEMENT EQUATIONS

Thus the element equations with $W_i = N_i$ are,

$$\int_{\Gamma^{(e)}} N_i \left(k\frac{\partial T}{\partial x} n_x + k\frac{\partial T}{\partial y} n_y \right) d\Gamma - \int_{\Omega^{(e)}} \left(\frac{\partial N_i}{\partial x} k \frac{\partial T}{\partial x} + \frac{\partial N_i}{\partial y} k \frac{\partial T}{\partial y} \right) d\Omega$$

$$+ \int_{\Omega^{(e)}} N_i \, Q \, d\Omega \quad = \quad 0 \qquad i = 1, 2, 3$$

Imposing the boundary conditions along edges, the element equations can be written in matrix form (total of 3 eqs.) as,

$$\int_{\Omega^{(e)}} \left(\left\{ \frac{\partial N}{\partial x} \right\} k \frac{\partial T}{\partial x} + \left\{ \frac{\partial N}{\partial y} \right\} k \frac{\partial T}{\partial y} \right) d\Omega$$

$$= \int_{\Gamma^{(e)}} \{N\} \left(k\frac{\partial T}{\partial x} n_x + k\frac{\partial T}{\partial y} n_y \right) d\Gamma + \int_{\Omega^{(e)}} \{N\} Q \, d\Omega + \int_{\Gamma^{(e)}} \{N\} q_s \, d\Gamma$$

DERIVATION OF ELEMENT EQUATIONS

Since we assume element temperature distribution,

$$T = T(x, y) = \underset{(1x3)}{\lfloor N \rfloor} \underset{(3x1)}{\{T\}}$$

$$\frac{\partial T}{\partial x} = \underset{(1x3)}{\left\lfloor \frac{\partial N}{\partial x} \right\rfloor} \underset{(3x1)}{\{T\}}$$

and

$$\frac{\partial T}{\partial y} = \underset{(1x3)}{\left\lfloor \frac{\partial N}{\partial y} \right\rfloor} \underset{(3x1)}{\{T\}}$$

Substitute to get the final form of element equations,

DERIVATION OF ELEMENT EQUATIONS

$$\underbrace{\int_{\Omega^{(e)}} \left(\underset{(3x1)}{\left\{ \frac{\partial N}{\partial x} \right\}} k \underset{(1x3)}{\left\lfloor \frac{\partial N}{\partial x} \right\rfloor} + \underset{(3x1)}{\left\{ \frac{\partial N}{\partial y} \right\}} k \underset{(1x3)}{\left\lfloor \frac{\partial N}{\partial y} \right\rfloor} \right) d\Omega}_{\underset{(3x3)}{[K_c]}} \underset{(3x1)}{\{T\}}$$

$$= \underbrace{\int_{\Gamma^{(e)}} \underset{(3x1)}{\{N\}} \left(k \frac{\partial T}{\partial x} n_x + k \frac{\partial T}{\partial y} n_y \right) d\Gamma}_{\underset{(3x1)}{\{Q_c\}}} + \underbrace{\int_{\Omega^{(e)}} \underset{(3x1)}{\{N\}} Q d\Omega}_{\underset{(3x1)}{\{Q_Q\}}} + \underbrace{\int_{S_2} \underset{(3x1)}{\{N\}} q_s d\Gamma}_{\underset{(3x1)}{\{Q_q\}}}$$

Or, in short (details given in text):

$$\lfloor K_c \rfloor \{T\} = \{Q_c\} + \{Q_Q\} + \{Q_q\}$$

FINITE ELEMENT MATRICES

Element matrices for triangle can be derived in closed-form. Since,

$$N_i = \frac{1}{2A}(a_i + b_i x + c_i y) \qquad i = 1, 2, 3$$

then $\dfrac{\partial N_i}{\partial x} = \dfrac{b_i}{2A}$ and $\dfrac{\partial N_i}{\partial y} = \dfrac{c_i}{2A}$

<u>Conduction matrix</u> Coefficients in the matrix are,

$$K_{ij} = \int_A k \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) t\, dx\, dy \qquad i, j = 1, 2, 3$$

For constant thermal conductivity k and thickness t, then

$$K_{ij} = k\, t \int_A \left(\frac{b_i}{2A} \frac{b_j}{2A} + \frac{c_i}{2A} \frac{c_j}{2A} \right) dx\, dy$$

FINITE ELEMENT MATRICES

Or, $K_{ij} = \frac{k\,t}{4A}(b_i\, b_j + c_i\, c_j)$ i, j = 1, 2, 3

$$\underset{(3x3)}{\lfloor K_c \rfloor} = k\,A\,t\ \underset{(3x2)}{[B]^T}\ \underset{(2x3)}{[B]} \quad \text{where} \quad [B] = \frac{1}{2A}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

<u>Load vector due to heat generation</u> Coefficients are,

$$Q_i = \int_A N_i\, Q\, t\, dx\, dy \qquad i = 1, 2, 3$$

For constant Q and t over element, integrate to get,

$$Q_i = \frac{Q\,A\,t}{3}$$

FINITE ELEMENT MATRICES

Or, in matrix form: $\{Q_Q\} = \frac{Q\,A\,t}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$

<u>Load vector due to edge heating</u>

$\{Q_q\} = \int_{s_2} \{N\}\, q_s\, d\Gamma$ E.g., $\{Q_q\} = \frac{q_s\, t\, \ell_{12}}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$

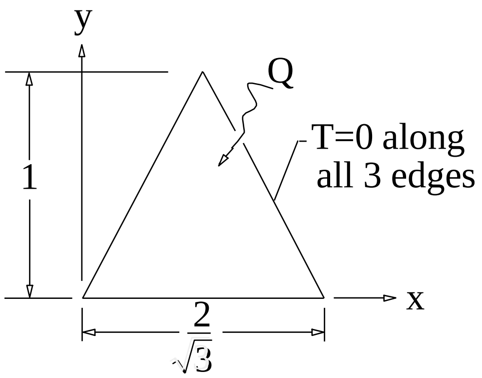
having nodes 1 and 2 on the edge.

<u>Load vector for conduction</u>

$$\{Q_c\} = \int_{\Gamma^{(e)}} \{N\}\left(k\frac{\partial T}{\partial x} n_x + k\frac{\partial T}{\partial y} n_y \right) d\Gamma$$

TWO-DIMENSIONAL HEAT TRANSFER

Example Determine temperature distribution in a triangular plate with internal heat generation using 3 finite elements.

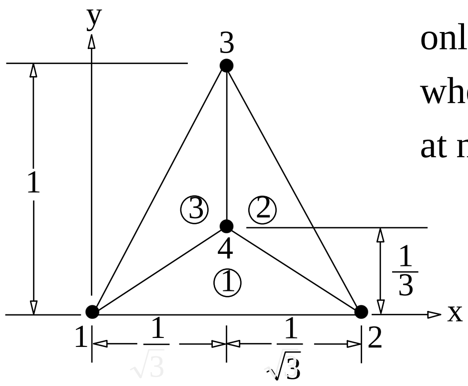


Note: This example has exact temperature solution as,

$$\bar{T}(x, y) = \frac{Q}{4k}(y - 2 + \sqrt{3}\,x)(y - \sqrt{3}\,x)\,y$$

TWO-DIMENSIONAL HEAT TRANSFER

we first discretize model into 3 elements. Here we have only one unknown at node 4 where as temperatures are zero at nodes 1, 2, and 3.



Typical element eqs. are:

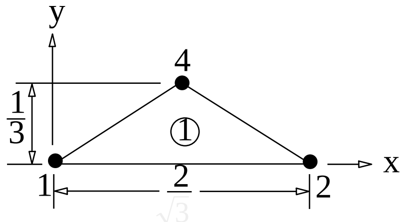
$$\lfloor K_c \rfloor \{T\} = \{Q_c\} + \{Q_Q\}$$

e.g., for element no. 1,

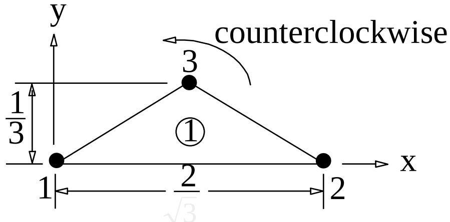
$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} Q_{c_1} \\ Q_{c_2} \\ Q_{c_3} \end{Bmatrix} + \frac{QAt}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

TWO-DIMENSIONAL HEAT TRANSFER

Computation of element matrices:



Actual node numbers



Node numbers in derivation

Here,

$$x_1 = 0 \qquad y_1 = 0 \qquad b_1 = -\frac{1}{3} \qquad c_1 = -\frac{1}{\sqrt{3}}$$

$$x_2 = \frac{2}{\sqrt{3}} \qquad y_2 = 0 \qquad b_2 = \frac{1}{3} \qquad c_2 = -\frac{1}{\sqrt{3}}$$

$$x_3 = \frac{1}{\sqrt{3}} \qquad y_3 = \frac{1}{3} \qquad b_3 = 0 \qquad c_3 = \frac{2}{\sqrt{3}}$$

with the element area $A = 1/3\sqrt{3}$.

TWO-DIMENSIONAL HEAT TRANSFER

With these coefficients b_i and c_i, i = 1, 2, 3, element matrices can be determined. For example the coefficient K_{23} in the conduction matrix is,

$$K_{23} = \frac{k\,t}{4A}\left(b_2\, b_3 + c_2\, c_3\right)$$

$$= \frac{k\,t}{4\left(\frac{1}{3\sqrt{3}}\right)}\left(\left(\frac{1}{3}\right)(0) + \left(-\frac{1}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)\right) = k\,t\left(-\frac{3}{2\sqrt{3}}\right)$$

Thus, element equations for element no. ① are,

$$\frac{k\,t}{2\sqrt{3}}\begin{bmatrix} 2 & 1 & -3 \\ 1 & 2 & -3 \\ -3 & -3 & 6 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} (Q_{c_1})_1 \\ (Q_{c_2})_1 \\ (Q_{c_4})_1 \end{Bmatrix} + \frac{Q\,t}{9\sqrt{3}}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

TWO-DIMENSIONAL HEAT TRANSFER

Similary, element eqs. for element no. ② are

$$\frac{k\,t}{2\sqrt{3}}\begin{bmatrix} 2 & 1 & -3 \\ 1 & 2 & -3 \\ -3 & -3 & 6 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} (Q_{c_2})_2 \\ (Q_{c_3})_2 \\ (Q_{c_4})_2 \end{Bmatrix} + \frac{Q\,t}{9\sqrt{3}}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

and for element no. ③ are,

$$\frac{k\,t}{2\sqrt{3}}\begin{bmatrix} 2 & 1 & -3 \\ 1 & 2 & -3 \\ -3 & -3 & 6 \end{bmatrix}\begin{Bmatrix} T_3 \\ T_1 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} (Q_{c_3})_3 \\ (Q_{c_1})_3 \\ (Q_{c_4})_3 \end{Bmatrix} + \frac{Q\,t}{9\sqrt{3}}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Then assemble element equations. from these 3 elements to get,

TWO-DIMENSIONAL HEAT TRANSFER

$$\frac{k\,t}{2\sqrt{3}}\begin{bmatrix} 2+2 & 1 & 1 & -3-3 \\ 1 & 2+2 & 1 & -3-3 \\ 1 & 1 & 2+2 & -3-3 \\ -3-3 & -3-3 & -3-3 & 6+6+6 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix}$$

$$= \begin{Bmatrix} (Q_{c_1})_1 + (Q_{c_1})_3 \\ (Q_{c_2})_1 + (Q_{c_2})_2 \\ (Q_{c_3})_2 + (Q_{c_3})_3 \\ (Q_{c_4})_1 + (Q_{c_4})_2 + (Q_{c_4})_3 \end{Bmatrix} + \frac{Q\,t}{9\sqrt{3}}\begin{Bmatrix} 1+1 \\ 1+1 \\ 1+1 \\ 1+1+1 \end{Bmatrix}$$

TWO-DIMENSIONAL HEAT TRANSFER

Apply boundary conditions of $T_1 = T_2 = T_3 = 0$, the system eqs. become,

$$\frac{k\,t}{2\sqrt{3}}\begin{bmatrix} 4 & 1 & 1 & -6 \\ 1 & 4 & 1 & -6 \\ 1 & 1 & 4 & -6 \\ -6 & -6 & -6 & 18 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ 0 \end{Bmatrix} + \frac{Q\,t}{9\sqrt{3}}\begin{Bmatrix} 2 \\ 2 \\ 2 \\ 3 \end{Bmatrix}$$

Then use the last equation to solve for temperature at node 4,

$$\frac{k\,t}{2\sqrt{3}}(0 + 0 + 0 + 18\,T_4) = 0 + \frac{Q\,t}{9\sqrt{3}}(3)$$

$$T_4 = \frac{1}{27}\frac{Q}{k}$$

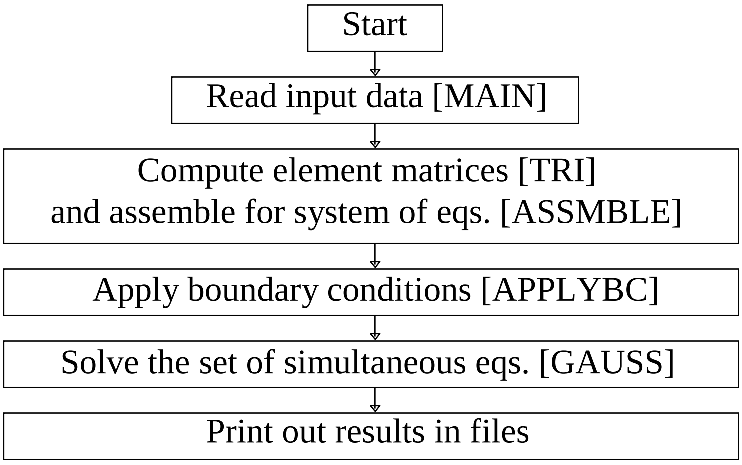
and then can use the first 3 equations to solve for heat fluxes at nodes,

$$Q_1 = Q_2 = Q_3 = -\frac{6\,k\,t}{2\sqrt{3}}T_4 - \frac{2\,Q\,t}{9\sqrt{3}} = -\frac{Q\,t}{3\sqrt{3}}$$

COMPUTER PROGRAMMING

- Same procedure described in example can be used in the development of finite element computer program directly.
- Listing of sample finite element program provided in following pages (as short as only 6 A-4 pages).
- Program is applicable to arbitrary geometry.
- Program can be modified to solve other problems.
- By understanding this short program -- won't be afraid of "black box" anymore.

COMPUTER PROGRAMMING FLOW-CHART



Note: Names in the square brackets [ ] represent the subroutine names in the finite element program illustrated.

FINITE ELEMENT PROGRAM LISTING

Example in FORTRAN

```
C     PROGRAM  FINITE
C
C     A FINITE ELEMENT COMPUTER PROGRAM FOR SOLVING PARTIAL
C     DIFFERENTIAL EQUATION IN THE FORM OF POISSON'S EQUATION
C     FOR TWO-DIMENSIONAL STEADY-STATE HEAT CONDUCTION WITH
C     INTERNAL HEAT GENERATION.
C                                       DR. PRAMOTE DECHAUMPHAI
C                                       FACULTY OF ENGINEERING
C                                       CHULALONGKORN UNIVERSITY
C
C     THE VALUES DECLARED IN THE PARAMETER STATEMENT BELOW SHOULD
C     BE ADJUSTED ACCORDING TO THE SIZE OF THE PROBLEMS AND TYPES
C     OF COMPUTERS:
C         MXPOI = MAXIMUM NUMBER OF NODES IN THE MODEL
C         MXELE = MAXIMUM NUMBER OF ELEMENTS IN THE MODEL
C
      PARAMETER (MXPOI=150, MXELE=300)
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION  COORD(MXPOI,2), TEMP(MXPOI), TEXT(20)
      DIMENSION  SYSK(MXPOI,MXPOI), SYSQ(MXPOI), QELE(MXELE)
      CHARACTER*20  NAME1, NAME2
C
      INTEGER  INTMAT(MXELE,3), IBC(MXPOI)
C
   10 WRITE(6,20)
   20 FORMAT(/, ' PLEASE ENTER THE INPUT FILE NAME:')
      READ(5, '(A)', ERR=10)  NAME1
      OPEN(UNIT=7, FILE=NAME1, STATUS='OLD', ERR=10)
C
C     READ TITLE OF COMPUTATION:
C
      READ(7,*)  NLINES
      DO 100  ILINE=1,NLINES
      READ(7,1)  TEXT
    1 FORMAT(20A4)
  100 CONTINUE
C
C     READ INPUT DATA:
C
      READ(7,1)  TEXT
      READ(7,*)  NPOIN, NELEM
      IF(NPOIN.GT.MXPOI)  WRITE(6,110)  NPOIN
  110 FORMAT(/,' PLEASE INCREASE THE PARAMETER MXPOI TO ', I5)
      IF(NPOIN.GT.MXPOI)  STOP
      IF(NELEM.GT.MXELE)  WRITE(6,120)  NELEM
  120 FORMAT(/,' PLEASE INCREASE THE PARAMETER MXELE TO ', I5)
      IF(NELEM.GT.MXELE)  STOP
      READ(7,1)  TEXT
      READ(7,*)  TK, THICK
      READ(7,1)  TEXT
      DO 130  IP=1,NPOIN
      READ(7,*)  I, IBC(I), (COORD(I,K), K=1,2), TEMP(I)
      IF(I.NE.IP)  WRITE(6,135)  IP
  135 FORMAT(/, ' NODE NO.', I5, ' IN DATA FILE IS MISSING')
      IF(I.NE.IP)  STOP
  130 CONTINUE
      IQ = 0
      READ(7,1)  TEXT
      DO 140  IE=1,NELEM
      READ(7,*)  I, (INTMAT(I,J), J=1,3), QELE(I)
      IF(I.NE.IE)  WRITE(6,150)  IE
  150 FORMAT(/, ' ELEMENT NO.', I5, ' IN DATA FILE IS MISSING')
      IF(I.NE.IE)  STOP
      IF(QELE(I).NE.0.)  IQ = 1
  140 CONTINUE
      WRITE(6,160)
  160 FORMAT(/,' THE F.E. MODEL INCLUDES THE FOLLOWING',
     *         ' HEAT TRANSFER MODE(S):',
     *      /,'    --  HEAT CONDUCTION               ')
      IF(IQ.EQ.1)  WRITE(6,170)
  170 FORMAT(  '    --  INTERNAL HEAT GENERATION      ')
C
      NEQ  = NPOIN
      DO 180  I=1,NEQ
      SYSQ(I) = 0.
  180 CONTINUE
      DO 190  I=1,NEQ
      DO 190  J=1,NEQ
      SYSK(I,J) = 0.
  190 CONTINUE
C
      WRITE(6,200)  NPOIN, NELEM
  200 FORMAT(/,' *** THE FINITE ELEMENT MODEL CONSISTS OF', I5,
     *         ' NODES AND', I5,' ELEMENTS ***')
C
C     ESTABLISH ALL ELEMENT MATRICES AND ASSEMBLE THEM TO FORM
C     FORM UP SYSTEM EQUATIONS
C
      WRITE(6,210)
  210 FORMAT(/,' *** ESTABLISHING ELEMENT MATRICES AND',
     *         ' ASSEMBLING ELEMENT EQUATIONS ***'     )
      CALL TRI(NELEM, INTMAT, COORD,    TK, QELE, THICK,
     *          SYSK,   SYSQ, MXPOI, MXELE             )
C
      WRITE(6,220)
  220 FORMAT(/,' *** APPLYING BOUNDARY CONDITIONS OF NODAL',
     *         ' TEMERATURES ***'                          )
      CALL APPLYBC(NPOIN, IBC, TEMP, SYSK, SYSQ, MXPOI)
C
      WRITE(6,230)
  230 FORMAT(/,' *** SOLVING A SET OF SIMULTANEOUS EQUATIONS',
     *         ' FOR TEMPERATURE SOLUTIONS ***'             )
      WRITE(6,240)  NEQ
  240 FORMAT(5X,'( TOTAL OF', I5,' EQUATIONS TO BE SOLVED )')
      CALL GAUSS(NEQ, SYSK, SYSQ, TEMP, MXPOI)
C
C     PRINT OUT NODAL TEMPERATURE SOLUTIONS:
C
  250 WRITE(6,260)
  260 FORMAT(/, ' PLEASE ENTER FILE NAME FOR TEMPERATURE'
     *          ' SOLUTIONS:'                          )
      READ(5, '(A)', ERR=250)  NAME2
      OPEN(UNIT=8, FILE=NAME2, STATUS='NEW', ERR=250)
      WRITE(8,270)  NPOIN
  270 FORMAT('  NODAL TEMPERATURE SOLUTIONS [', I5,']:',
     *      //, 2X, 'NODE', 3X, 'TEMPERATURE', /      )
      DO 280  IP=1,NPOIN
      WRITE(8,290)  IP, TEMP(IP)
  290 FORMAT(I6, E14.6)
  280 CONTINUE
C
      STOP
      END
```

FINITE ELEMENT PROGRAM LISTING

```
      SUBROUTINE APPLYBC(NPOIN, IBC, TEMP, SYSK, SYSQ, MXPOI)
C
C     APPLY TEMPERATURE BOUNDARY CONDITIONS WITH CONDITION CODES OF:
C           0 = FREE TO CHANGE (TO BE COMPUTED)
C           1 = FIXED AS SPECIFIED
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION  SYSK(MXPOI,MXPOI), SYSQ(MXPOI), TEMP(MXPOI)
C
      INTEGER  IBC(MXPOI)
C
      DO 100  IEQ=1,NPOIN
      IF(IBC(IEQ).EQ.0)  GO TO 100
C
      DO 200  IR=1,NPOIN
      IF(IR.EQ.IEQ)  GO TO 200
      SYSQ(IR) = SYSQ(IR) - SYSK(IR,IEQ)*TEMP(IEQ)
      SYSK(IR,IEQ) = 0.
  200 CONTINUE
C
      DO 300  IC=1,NPOIN
      SYSK(IEQ,IC) = 0.
  300 CONTINUE
      SYSK(IEQ,IEQ) = 1.
      SYSQ(IEQ) = TEMP(IEQ)
C
  100 CONTINUE
C
      RETURN
      END
C
C---------------------------------------------------------------------
C
      SUBROUTINE ASSMBLE(   IE, INTMAT, AKC, QQ, SYSK, SYSQ,
     *                   MXPOI,  MXELE                        )
C
C     ASSEMBLE ELEMENT EQUATIONS INTO SYSTEM EQUATIONS
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION  AKC(3,3), QQ(3)
      DIMENSION  SYSK(MXPOI,MXPOI), SYSQ(MXPOI)
C
      INTEGER  INTMAT(MXELE,3)
C
      NNODE = 3
C
      DO 100  IR=1,NNODE
      DO 200  IC=1,NNODE
      IROW = INTMAT(IE,IR)
      ICOL = INTMAT(IE,IC)
      SYSK(IROW,ICOL) = SYSK(IROW,ICOL) + AKC(IR,IC)
  200 CONTINUE
      SYSQ(IROW) = SYSQ(IROW) + QQ(IR)
  100 CONTINUE
C
      RETURN
      END
C
C---------------------------------------------------------------------
C
      SUBROUTINE GAUSS(N, A, B, X, MXPOI)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION  A(MXPOI,MXPOI), B(MXPOI), X(MXPOI)
C
C     PERFORM SCALING:
C
      CALL SCALE(N, A, B, MXPOI)
C
C     FORWARD ELIMINATION:
C
C     PERFORM ACCORDING TO ORDER OF 'PRIME' FROM 1 TO N-1:
C
      DO 100  IP=1,N-1
C
C     PERFORM PARTIAL PIVOTING:
C
      CALL PIVOT(N, A, B, MXPOI, IP)
C
C     LOOP OVER EACH EQUATION STARTING FROM THE ONE THAT CORRESPONDS
C     WITH THE ORDER OF 'PRIME' PLUS ONE:
C
      DO 200  IE=IP+1,N
      RATIO = A(IE,IP)/A(IP,IP)
C
C     COMPUTE NEW COEFFICIENTS OF THE EQUATION CONSIDERED:
C
      DO 300  IC=IP+1,N
      A(IE,IC) = A(IE,IC) - RATIO*A(IP,IC)
  300 CONTINUE
      B(IE) = B(IE) - RATIO*B(IP)
  200 CONTINUE
C
C     SET COEFFICIENTS ON LOWER LEFT PORTION TO ZERO:
C
      DO 400  IE=IP+1,N
      A(IE,IP) = 0.
  400 CONTINUE
  100 CONTINUE
C
C     BACK SUBSTITUTION:
C
C     COMPUTE SOLUTION OF THE LAST EQUATION:
C
      X(N) = B(N)/A(N,N)
C
C     THEN COMPUTE SOLUTIONS FROM EQUATION N-1 TO 1:
C
      DO 500  IE=N-1,1,-1
      SUM = 0.
      DO 600  IC=IE+1,N
      SUM = SUM + A(IE,IC)*X(IC)
  600 CONTINUE
      X(IE) = (B(IE) - SUM)/A(IE,IE)
  500 CONTINUE
      RETURN
      END
C
C---------------------------------------------------------------------
C
      SUBROUTINE PIVOT(N, A, B, MXPOI, IP)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION  A(MXPOI,MXPOI), B(MXPOI)
C
C     PERFORM PARTIAL PIVOTING:
C
      JP = IP
      BIG = ABS(A(IP,IP))
      DO 10  I=IP+1,N
      AMAX = ABS(A(I,IP))
      IF(AMAX.GT.BIG)  THEN
         BIG = AMAX
         JP  = I
      ENDIF
   10 CONTINUE
```


FINITE ELEMENT PROGRAM LISTING

```
      IF(AMAX.GT.BIG)  THEN
         BIG = AMAX
         JP  = I
      ENDIF
   10 CONTINUE
         IF(JP.NE.IP)  THEN
      DO 20  J=IP,N
      DUMY    = A(JP,J)
      A(JP,J) = A(IP,J)
      A(IP,J) = DUMY
   20 CONTINUE
      DUMY  = B(JP)
      B(JP) = B(IP)
      B(IP) = DUMY
         ENDIF
      RETURN
      END
C
C---------------------------------------------------------------------
C
      SUBROUTINE SCALE(N, A, B, MXPOI)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION  A(MXPOI,MXPOI), B(MXPOI)
C
C     PERFORM SCALING:
C
      DO 10  IE=1,N
      BIG = ABS(A(IE,1))
      DO 20  IC=2,N
      AMAX = ABS(A(IE,IC))
      IF(AMAX.GT.BIG)  BIG = AMAX
   20 CONTINUE
      DO 30  IC=1,N
      A(IE,IC) = A(IE,IC)/BIG
   30 CONTINUE
      B(IE) = B(IE)/BIG
   10 CONTINUE
      RETURN
      END
C
C---------------------------------------------------------------------
C
      SUBROUTINE TRI(NELEM, INTMAT, COORD,    TK, QELE, THICK,
     *               SYSK,   SYSQ, MXPOI, MXELE              )
C
C     ESTABLISH ALL ELEMENT MATRICES AND ASSEMBLE THEM TO FORM
C     UP SYSTEM EQUATIONS
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION  COORD(MXPOI,2), SYSK(MXPOI,MXPOI), SYSQ(MXPOI)
      DIMENSION  QELE(MXELE)
      DIMENSION  AKC(3,3), QQ(3), B(2,3), BT(3,2)
C
      INTEGER  INTMAT(MXELE,3)
C
C     LOOP OVER THE NUMBER OF ELEMENTS:
C
      DO 500  IE=1,NELEM
C
C     FIND ELEMENT LOCAL COORDINATES:
C
      II = INTMAT(IE,1)
      JJ = INTMAT(IE,2)
      KK = INTMAT(IE,3)
C
      XG1 = COORD(II,1)
      XG2 = COORD(JJ,1)
      XG3 = COORD(KK,1)
      YG1 = COORD(II,2)
      YG2 = COORD(JJ,2)
      YG3 = COORD(KK,2)
      AREA= 0.5*(XG2*(YG3-YG1) + XG1*(YG2-YG3) + XG3*(YG1-YG2))
      IF(AREA.LE.0.)  WRITE(6,5)  IE
    5 FORMAT(/,'  !!! ERROR !!!  ELEMENT NO.', I5,
     *          ' HAS NEGATIVE OR ZERO AREA ', /,
     *          '  --- CHECK F.E. MODEL FOR NODAL COORDINATES',
     *          ' AND ELEMENT NODAL CONNECTIONS ---'          )
      IF(AREA.LE.0.)  STOP
C
      B1 = YG2 - YG3
      B2 = YG3 - YG1
      B3 = YG1 - YG2
      C1 = XG3 - XG2
      C2 = XG1 - XG3
      C3 = XG2 - XG1
C
      DO 10  I=1,2
      DO 10  J=1,3
      B(I,J) = 0.
   10 CONTINUE
C
      B(1,1) = B1
      B(1,2) = B2
      B(1,3) = B3
      B(2,1) = C1
      B(2,2) = C2
      B(2,3) = C3
C
      DO 20  I=1,2
      DO 30  J=1,3
      B(I,J)  = B(I,J)/(2.*AREA)
      BT(J,I) = B(I,J)
   30 CONTINUE
   20 CONTINUE
C
C     ELEMENT CONDUCTION MATRIX:
C
      DO 100  I=1,3
      DO 100  J=1,3
      AKC(I,J) = 0.
      DO 110  K=1,2
      AKC(I,J) = AKC(I,J) + BT(I,K)*B(K,J)
  110 CONTINUE
      AKC(I,J) = TK*AREA*THICK*AKC(I,J)
  100 CONTINUE
C
C     ELEMENT LOAD VECTOR DUE TO INTERNAL HEAT GENERATION:
C
      FAC = QELE(IE)*AREA*THICK/3.
      DO 200  I=1,3
      QQ(I) = FAC
  200 CONTINUE
C
C     ASSEMBLE THESE ELEMENT MATRICES TO FORM SYSTEM EQUATIONS:
C
      CALL ASSMBLE(   IE, INTMAT,   AKC, QQ, SYSK, SYSQ,
     *              MXPOI,  MXELE                        )
C
  500 CONTINUE
C
      RETURN
      END
```

TYPICAL INPUT & OUTPUT DATA

Input data

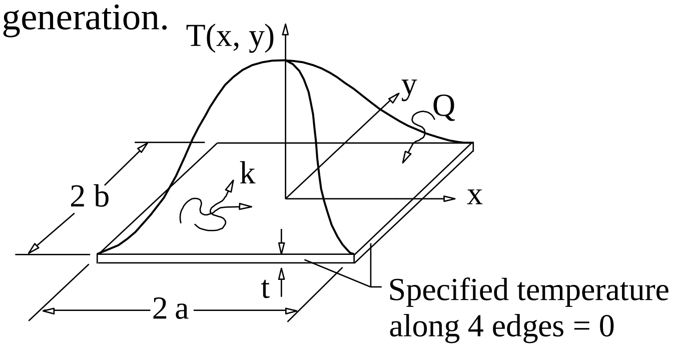
```
2
TRIANGULAR PLATE WITH INTERNAL HEAT GENERATION.
CRUDE MESH WITH 4 NODES AND 3 ELEMENTS.
NPOIN   NELEM
    4       3
   TK   THICK
   1.      .1
NODAL BOUNDARY CONDITIONS AND COORDINATES [4]:
    1     1     0.00000    0.00000    0.
    2     1     1.15470    0.00000    0.
    3     1     0.57735    1.00000    0.
    4     0     0.57735    0.33333    0.
ELEMENT NODAL CONNECTIONS AND HEAT GEN. [3]:
    1    1    2    4    1.
    2    2    3    4    1.
    3    3    1    4    1.
```

Output result

```
NODE   TEMPERATURE
   1   .000000E+00
   2   .000000E+00
   3   .000000E+00
   4   .370370E-01
```


RECTANGULAR ELEMENT

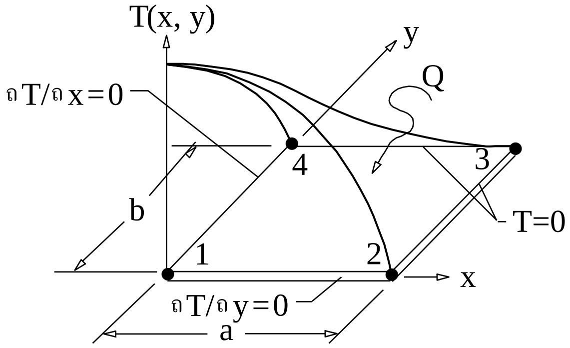
Example Two-dimensional steady-state heat conduction in a rectangular plate with internal heat generation.



Due to symmetry, only a quarter of the plate can be modeled. Use one 4-node rectangular element to compute temperature at the plate center (x=y=0).

RECTANGULAR ELEMENT

By using only <u>one</u> rectangular element for quarter of the plate, some edge conditions must be provided,

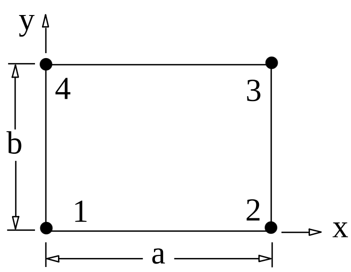


Here $T_2 = T_3 = T_4 = 0$

Want $T_1 = ?$

RECTANGULAR ELEMENT

<u>Element interpolation function</u>:



Assume element temperature,

$T(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$

which is in form of bilinear distribution where α_i , i = 1, 4 are constants that can be determined from the conditions,

$$T(0,0) = T_1 = \alpha_1 + 0 + 0 + 0$$
$$T(a,0) = T_2 = \alpha_1 + \alpha_2 a + 0 + 0$$
$$T(a,b) = T_3 = \alpha_1 + \alpha_2 a + \alpha_3 b + \alpha_4 ab$$
$$T(0,b) = T_4 = \alpha_1 + 0 + \alpha_3 b + 0$$

RECTANGULAR ELEMENT

Solve for $\alpha_i, i = 1, 4$ to get,

$$\alpha_1 = T_1 \qquad \alpha_3 = \frac{T_4 - T_1}{b}$$

$$\alpha_2 = \frac{T_2 - T_1}{a} \qquad \alpha_4 = \frac{T_1 - T_2 + T_3 - T_4}{ab}$$

Then, substitute back and rearrange terms to get element temperature distribution in the form,

$$T = \left\lfloor \underbrace{(1-\frac{x}{a})(1-\frac{y}{b})}_{N_1} \quad \underbrace{\frac{x}{a}(1-\frac{y}{b})}_{N_2} \quad \underbrace{\frac{x}{a}\frac{y}{b}}_{N_3} \quad \underbrace{\frac{y}{b}(1-\frac{x}{a})}_{N_4} \right\rfloor \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix}$$

$$= \underset{(1x4)}{\lfloor N(x,y) \rfloor} \; \underset{(4x1)}{\{T\}}$$

where $\lfloor N(x,y) \rfloor$ is the element interpolation function matrix.

PLATE WITH INTERNAL HEAT GEN.

Since the corresponding FE eqs. for this problem are,

$$\lfloor K_c \rfloor \{T\} = \{Q_c\} + \{Q_Q\}$$

where $\lfloor K_c \rfloor$ is the conduction matrix,

$$[K_c] = \int_{\Omega^{(e)}} k \left(\begin{Bmatrix} \frac{\partial N}{\partial x} \end{Bmatrix} \left\lfloor \frac{\partial N}{\partial x} \right\rfloor + \begin{Bmatrix} \frac{\partial N}{\partial y} \end{Bmatrix} \left\lfloor \frac{\partial N}{\partial y} \right\rfloor \right) d\Omega$$

Here,

$$\left\lfloor \frac{\partial N}{\partial x} \right\rfloor = \left\lfloor -\frac{1}{a}(1-\frac{y}{b}) \quad \frac{1}{a}(1-\frac{y}{b}) \quad \frac{1}{a}\frac{y}{b} \quad -\frac{1}{a}\frac{y}{b} \right\rfloor$$

$$\left\lfloor \frac{\partial N}{\partial y} \right\rfloor = \left\lfloor -\frac{1}{b}(1-\frac{x}{a}) \quad -\frac{1}{b}\frac{x}{a} \quad \frac{1}{b}\frac{x}{a} \quad \frac{1}{b}(1-\frac{x}{a}) \right\rfloor$$

and $d\Omega = t\,dx\,dy$. Perform integration with the limits $x = 0 \rightarrow a$ and $y = 0 \rightarrow b$ to obtain,

FINITE ELEMENT EQUATIONS

Conduction matrix,

$$[K_c] = k\,t \begin{bmatrix} \frac{1}{3}\left(\frac{a}{b}+\frac{b}{a}\right) & \frac{1}{6}\left(\frac{a}{b}-\frac{2b}{a}\right) & \frac{1}{6}\left(-\frac{a}{b}-\frac{b}{a}\right) & \frac{1}{6}\left(-\frac{2a}{b}+\frac{b}{a}\right) \\ & \frac{1}{3}\left(\frac{a}{b}+\frac{b}{a}\right) & \frac{1}{6}\left(-\frac{2a}{b}+\frac{b}{a}\right) & \frac{1}{6}\left(-\frac{a}{b}-\frac{b}{a}\right) \\ & & \frac{1}{3}\left(\frac{a}{b}+\frac{b}{a}\right) & \frac{1}{6}\left(\frac{a}{b}-\frac{2b}{a}\right) \\ \text{sym} & & & \frac{1}{3}\left(\frac{a}{b}+\frac{b}{a}\right) \end{bmatrix}$$

Also, the load vector from internal heat generation,

$$\{Q_Q\} = \int_{\Omega^{(e)}} Q\,\{N\}\,d\Omega = Q\,a\,b\,t \begin{Bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{Bmatrix}$$

FINITE ELEMENT EQUATIONS

and the conduction vector,

$$\{Q_c\} = \int_{\Gamma^{(e)}} \{N\}\,k\,\left(\frac{\partial T}{\partial x} n_x + \frac{\partial T}{\partial y} n_y\right) d\Gamma = \begin{Bmatrix} Q_{c_1} \\ Q_{c_2} \\ Q_{c_3} \\ Q_{c_4} \end{Bmatrix}$$

Therefore, the final element eqs. can be written as,

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ & K_{22} & K_{23} & K_{24} \\ & & K_{33} & K_{34} \\ \text{Sym} & & & K_{44} \end{bmatrix} \begin{Bmatrix} T_1 = ? \\ T_2 = 0 \\ T_3 = 0 \\ T_4 = 0 \end{Bmatrix} = \begin{Bmatrix} Q_{c_1} \\ Q_{c_2} \\ Q_{c_3} \\ Q_{c_4} \end{Bmatrix} + Q\,a\,b\,t \begin{Bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{Bmatrix}$$

Then use the first eq. to solve for T_1,

PLATE WITH INTERNAL HEAT GEN.

$$K_{11}T_1 + 0 + 0 + 0 = Q_{c_1} + \frac{Q\,a\,b\,t}{4}$$

$$\frac{k\,t}{3}\left(\frac{a}{b} + \frac{b}{a}\right)T_1 = Q_{c_1} + \frac{Q\,a\,b\,t}{4}$$

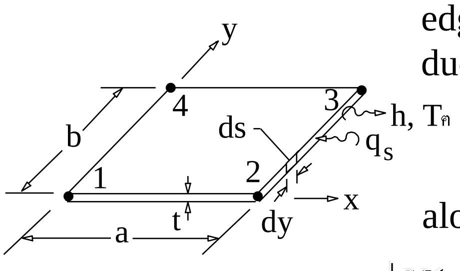
It can be shown that $Q_{c_1} = 0$ (no conduction across center of the plate), thus the temperature at the plate center is,

$$T_1 = \frac{3}{4}\,\frac{Q\,a\,b}{k}\,\frac{1}{\left(\frac{a}{b} + \frac{b}{a}\right)}$$

Note that if Q = k = a = b = 1, the exact solution is 0.295 where T_1 is 0.375. But if use 4 elements, T_1 becomes 0.315, i.e., solution is improved with more elements.

PLATE WITH EDGE HEAT TRANSFER

If the plate has specified heating and convection along edge 2-3. Since the load vector due to edge heating is,



$$\{Q_q\} = \int_{s_2^{(e)}} q_s \{N\}\, ds$$

along edge x = a, ds = tdy and

$$\lfloor N(x = a, y) \rfloor = \left\lfloor 0 \quad 1 - \frac{y}{b} \quad \frac{y}{b} \quad 0 \right\rfloor$$

Then,

$$\{Q_q\} = \int_0^b q_s \begin{Bmatrix} 0 \\ 1 - \frac{y}{b} \\ \frac{y}{b} \\ 0 \end{Bmatrix} t\,dy = q_s\,b\,t \begin{Bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{Bmatrix}$$

PLATE WITH EDGE HEAT TRANSFER

Similary, the convection matrix corresponding to convection heat transfer along edge 2-3 can be derived,

$$[K_h] = \int_{s_2^{(e)}} h \{N\} \lfloor N \rfloor ds$$

$$= \int_0 h \begin{Bmatrix} 0 \\ 1 - \frac{y}{b} \\ \frac{y}{b} \\ 0 \end{Bmatrix} \begin{bmatrix} 0 & 1 - \frac{y}{b} & \frac{y}{b} & 0 \end{bmatrix} t\, dy = h\, b\, t \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

PLATE WITH EDGE HEAT TRANSFER

with the convection load vector,

$$\{Q_h\} = \int_{s_2^{(e)}} h\, T_\infty \{N\} ds = \int_{s_2^{(e)}} h\, T_\infty \begin{Bmatrix} 0 \\ 1 - \frac{y}{b} \\ \frac{y}{b} \\ 0 \end{Bmatrix} t\, dy$$

$$= h\, b\, t\, T_\infty \begin{Bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{Bmatrix}$$