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FINITE ELEMENT METHOD FOR ELASTICITY PROBLEMS

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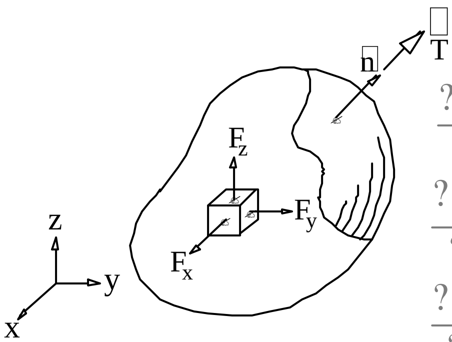
EQUILIBRIUM EQUATIONS



$$\frac{??_x}{?x} + \frac{??_{xy}}{?y} + \frac{??_{xz}}{?z} + F_x = 0$$

$$\frac{??_{xy}}{?x} + \frac{??_y}{?y} + \frac{??_{yz}}{?z} + F_y = 0$$

$$\frac{??_{xz}}{?x} + \frac{??_{yz}}{?y} + \frac{??_z}{?z} + F_z = 0$$

$?_x, ?_y, ?_z$ = Normal stresses in x, y, z directions

$?_{xy}, ?_{xz}, ?_{yz}$ = Shearing stresses

F_x, F_y, F_z = Body forces in x, y, z directions

BOUNDARY CONDITIONS

Typical boundary conditions include:

1. Specified displacement components, u, v, w.
2. Specified surface traction,

$$\vec{T} = T_x\hat{i} + T_y\hat{j} + T_z\hat{k}$$

where T_x, T_y, T_z are the tractions in x, y, z directions defined by,

$$?T_x? = ??_x \quad ?_{xy} \quad ?_{xz}? ?n_x?$$
$$?T_y? = ??_{xy} \quad ?_y \quad ?_{yz}? ?n_y?$$
$$?T_z? \quad ??_{xz} \quad ?_{yz} \quad ?_z? ?n_z?$$

and n_x, n_y, n_z are the direction cosines of the unit normal vector,

$$\hat{n} = n_x\hat{i} + n_y\hat{j} + n_z\hat{k}$$

STRESS-STRAIN RELATIONS

$$??? = ?C???-?_0?$$

(6x1) (6x6) (6x1)

where the stress vector contains stress components,

$$???^T = ??_x \quad ?_y \quad ?_z \quad ?_{xy} \quad ?_{yz} \quad ?_{xz}?$$

and the strain vector contains strain components,

$$???^T = ??_x \quad ?_y \quad ?_z \quad ?_{xy} \quad ?_{yz} \quad ?_{xz}?$$

and $??_0?$ is the vector of initial strains, such as thermal strain,

$$??_0?^T = ??? T \quad ??T \quad ??T \quad 0 \quad 0 \quad 0?$$

where ? is the coefficient of thermal expansion, and

$$?T = T(x, y, z) - T_0$$

STRESS-STRAIN RELATIONS

$$? T = T(x, y, z) - T_0$$

T(x, y, z) is the temperature at any x, y, z location and T_0 is the reference temperature for zero stress.

The material elasticity matrix is defined as,

$$?C? = \frac{E}{(1+?)(1-2?)} \begin{bmatrix} 1-? & ? & ? & 0 & 0 & 0 \\ ? & 1-? & ? & 0 & 0 & 0 \\ ? & ? & 1-? & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2?)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2?)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2?)/2 \end{bmatrix}$$

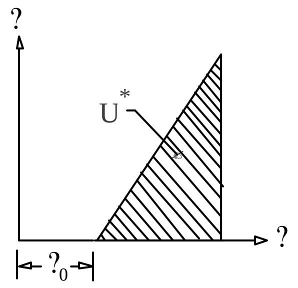
where E is the modulus of elasticity and ? is the Poisson's ratio.

VARIATIONAL FUNCTION

Total potential energy,

$$J = U^* + V^*$$

where U^* is the internal strain energy. From the figure,



$$U^* = \frac{1}{2} \int_V ?? - ?_0 ??? ? dV$$

Substitute stress-strain relations,

$$U^* = \frac{1}{2} \int_V ?? - ?_0 ? ?C? ?? - ?_0 ? dV$$

Expand to get,

$$U^* = \frac{1}{2} \int_V ???C???dV - \int_V ???C???_0 ?dV + \frac{1}{2} \int_V ??_0 ?C???_0 ?dV$$

VARIATIONAL FUNCTION

V^* is the potential energy from the body force, $\vec{F}$, and the surface traction, $\vec{T}$,

$$V^* = -\int_V (F_x u + F_y v + F_z w)\,dV - \int_S (T_x u + T_y v + T_z w)\,dS$$

$$= -\int_V \lfloor u \;\; v \;\; w \rfloor \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} dV - \int_S \lfloor u \;\; v \;\; w \rfloor \begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix} dS$$

$$= -\int_V \underset{(1x3)(3x1)}{\{\bar{u}\}^T\{F\}}\,dV - \int_S \underset{(1x3)(3x1)}{\{\bar{u}\}^T\{T\}}\,dS$$

where $\{\bar{u}\}$ is the vector that contains the displacements u, v, w in the x, y, z directions, respectively.

VARIATIONAL FUNCTION

Thus, the total potential energy is,

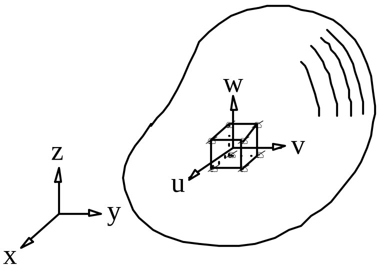
$$J = \frac{1}{2}\int_V \lfloor ? \rfloor [C] \{?\}\,dV - \int_V \lfloor ? \rfloor [C] \{?_0\}\,dV$$

$$+ \frac{1}{2}\int_V \lfloor ?_0 \rfloor [C] \{?_0\}\,dV - \int_V \{\bar{u}\}^T\{F\}\,dV - \int_S \{\bar{u}\}^T\{T\}\,dS$$

Depending on the element type selected, the corresponding element equations can be derived by performing minimization on the total potential energy statement above.

FINITE ELEMENT EQUATIONS

Consider three-dimensional 8-node brick element, the displacement components are,



$$u(x, y, z) = \lfloor N(x, y, z) \rfloor \{u\}$$
(1x8) (8x1)

$$v(x, y, z) = \lfloor N(x, y, z) \rfloor \{v\}$$
(1x8) (8x1)

$$w(x, y, z) = \lfloor N(x, y, z) \rfloor \{w\}$$
(1x8) (8x1)

or, can be written together as,

$$\{\bar{u}\} = [N(x, y, z)]\{\delta\}$$
(3x1) (3x24) (24x1)

FINITE ELEMENT EQUATIONS

where $\{\bar{u}\}^T = \lfloor u \;\; v \;\; w \rfloor$

$\{\delta\}^T = \lfloor u_1 \;\; v_1 \;\; w_1 \;\; u_2 \;\; v_2 \;\; w_2 \;\ldots\ldots\; u_8 \;\; v_8 \;\; w_8 \rfloor$

Then the strain-displacement relations are,

$$\underset{(6x1)}{\{\varepsilon\}} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{Bmatrix} = \underset{(6x24)}{[B(x, y, z)]} \underset{(24x1)}{\{\delta\}}$$

FINITE ELEMENT EQUATIONS

where [B(x,y,z)] is called the strain-displacement matrix. Thus, the total potential energy becomes,

$$J = \frac{1}{2}\int_V \underset{(1x24)}{\{d\}^T} \underset{(24x6)}{[B]^T} \underset{(6x6)}{[C]} \underset{(6x24)}{[B]} \underset{(24x1)}{\{d\}} dV$$

$$- \int_V \underset{(1x24)}{\{d\}^T} \underset{(24x6)}{[B]^T} \underset{(6x6)}{[C]} \underset{(6x1)}{\{\varepsilon_0\}} dV + \frac{1}{2}\int_V \underset{(1x6)}{\{\varepsilon_0\}^T} \underset{(6x6)}{[C]} \underset{(6x1)}{\{\varepsilon_0\}} dV$$

$$- \int_V \underset{(1x24)}{\{d\}^T} \underset{(24x3)}{[N]^T} \underset{(3x1)}{\{F\}} dV - \int_S \underset{(1x24)}{\{d\}^T} \underset{(24x3)}{[N]^T} \underset{(3x1)}{\{T\}} dS$$

FINITE ELEMENT EQUATIONS

Or, in short,

$$J = \frac{1}{2}\{d\}^T[K]\{d\} - \{d\}^T\{F_0\} + \frac{1}{2}\int_V \{\varepsilon_0\}^T[C]\{\varepsilon_0\} dV$$

$$- \{d\}^T\{F_B\} - \{d\}^T\{F_t\}$$

where,

$$\underset{(24x24)}{[K]} = \int_V \underset{(24x6)}{[B]^T} \underset{(6x6)}{[C]} \underset{(6x24)}{[B]} dV$$

$$\underset{(24x1)}{\{F_0\}} = \int_V \underset{(24x6)}{[B]^T} \underset{(6x6)}{[C]} \underset{(6x1)}{\{\varepsilon_0\}} dV$$

$$\underset{(24x1)}{\{F_B\}} = \int_V \underset{(24x3)}{[N]^T} \underset{(3x1)}{\{F\}} dV$$

$$\underset{(24x1)}{\{F_t\}} = \int_S \underset{(24x3)}{[N]^T} \underset{(3x1)}{\{T\}} dS$$

FINITE ELEMENT EQUATIONS

Minimizing the total potential energy,

$$\frac{?\,J}{????} = 0$$

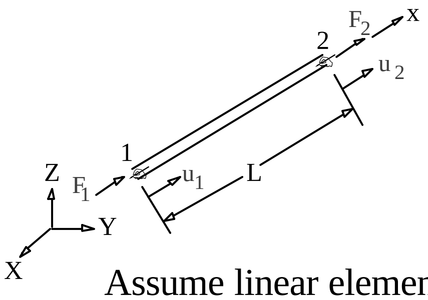
leads to F.E. eqs. in the form,

$$\underset{(24x24)}{?K?}\;\underset{(24x1)}{???} = \underset{(24x1)}{?F_0?} + \underset{(24x1)}{?F_B?} + \underset{(24x1)}{?F_t?}$$

After nodal displacements, ???, are solved, the element stresses can then be determined from,

$$\underset{(6x1)}{???} = \underset{(6x6)}{?C?}\;\underset{(6x24)}{?B?}\;\underset{(24x1)}{???} - \underset{(6x6)}{?C?}\;\underset{(6x1)}{??_0?}$$

TRUSS ELEMENT



Equilibrium equation,

$$\frac{??_x}{?x} + F_x = 0$$

Stress-strain relation,

$$?_x = E\left(\frac{?u}{?x} - ?_0\right)$$

Assume linear element distribution,

$$u(x) = ?N_1(x) \quad N_2(x)??\begin{matrix}u_1\\u_2\end{matrix}? = ?1 - \frac{x}{L} \quad \frac{x}{L}??\begin{matrix}u_1\\u_2\end{matrix}? = ?N????$$

then the element strain is

$$? = ?_x = \frac{?u}{?x} = ?-\frac{1}{L} \quad \frac{1}{L}??\begin{matrix}u_1\\u_2\end{matrix}? = ?B????$$

TRUSS ELEMENT

Knowing $[N]$ and $[B]$ matrices, all element matrices can be determined. Starting from the stiffness matrix,

$$[K] = \int_V [B]^T [C][B]\, dV$$

Here $[C] = E$ and $dV = A\, dx$,

$$[K] = \int_0^L \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} E \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} A\, dx$$

If the cross-sectional area A is constant, then,

$$[K] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

TRUSS ELEMENT

Load vector due to temperature is,

$$\{F_0\} = \int_V [B]^T [C]\{\varepsilon_0\}\, dV$$

where the thermal strain, $\varepsilon_0 = \alpha\, (T(x) - T_0)$

Thus,

$$\{F_0\} = \int_0^L \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} E\, \alpha\, (T(x) - T_0)\, A\, dx$$

If both the temperature T and cross-sectional area A are constant, then,

$$\{F_0\} = A\, E\, \alpha\, (T - T_0) \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

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HYPERSONIC WING STRUCTURE

<u>Reference</u>: Dechaumphai, P., "Improved Finite Element Methodology for Integrated Thermal Structural Analysis," <u>NASA CR</u> 3635, 1982.

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TWO-DIMENSIONAL ELEMENT

Equilibrium equations,

$$\frac{??_x}{?x} + \frac{??_{xy}}{?y} + F_x = 0$$

$$\frac{??_{xy}}{?x} + \frac{??_y}{?y} + F_y = 0$$

with boundary conditions of,

1. Specifying displacements u(x,y) and v(x,y) in x and y directions, respectively, or,

2. Specifying surface tractions,

$$\begin{matrix} ?T_x? \\ ? \quad ? \\ ?T_y? \end{matrix} = \begin{matrix} ??_x & ?_{xy} & ??n_x? \\ ? & & ?? \quad ? \\ ??_{xy} & ?_y & ??n_y? \end{matrix}$$

TWO-DIMENSIONAL ELEMENT

Stress-strain relations,

$$?? ? = ?C ??? - ?_0 ?$$

(3x1) (3x3) (3x1)

where the stress vector includes,

$$?? ?^T = ?? _x \quad ?_y \quad ?_{xy} ?$$

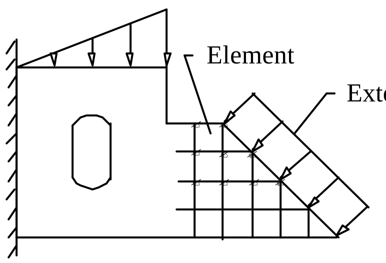
and the strain vector includes,

$$???^T = ??_x \quad ?_y \quad ?_{xy} ?$$

The pre-strain may be due to temperature change,

$$??_0 ?^T = ?? ?^T (T(x, y) - T_0)$$

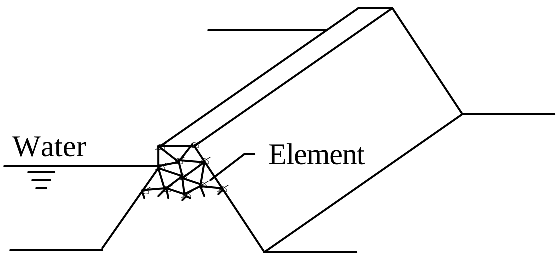
PLANE STRESS PROBLEM



For plane stress problem, the stress normal to the plane is negligible. The elasticity matrix and the thermal strain vector are,

$$?C? = \frac{E}{1 - ?^2} \begin{bmatrix} 1 & ? & 0 \\ ? & 1 & 0 \\ 0 & 0 & \frac{1-?}{2} \end{bmatrix} \qquad ?? ? = \begin{Bmatrix} ?? \\ ?? \\ 0 \end{Bmatrix}$$

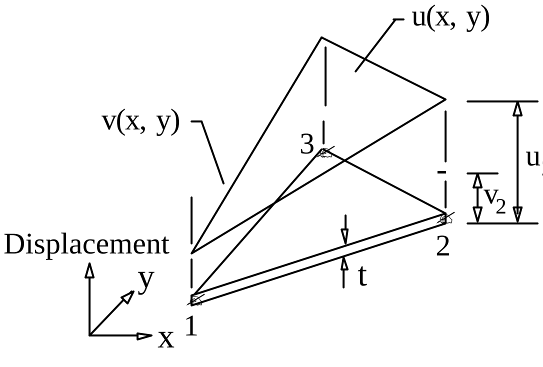
PLANE STRAIN PROBLEM



For plane strain problem, the strain normal to the plane is negligible. The elasticity matrix and the thermal strain vector are,

$$\{C\} = \frac{E}{(1+?)(1-2?)}\begin{bmatrix} 1-? & ? & 0 \\ ? & 1-? & 0 \\ 0 & 0 & \frac{1-2?}{2} \end{bmatrix} \qquad \{?\} = \begin{Bmatrix} ?? (1? ?) \\ ?? (1? ?) \\ 0 \end{Bmatrix}$$

TRIANGULAR ELEMENT



Assume "flat plane" displacement distributions over the element, i.e.,

TRIANGULAR ELEMENT

$$\underset{(2x1)}{\{\bar{u}\}} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$= \underset{(2x6)}{[N]} \; \underset{(6x1)}{\{d\}}$$

where N_i, i = 1, 2, 3 are the element interpolation functions.

TRIANGULAR ELEMENT

Then the strain-displacement relations are,

$$\underset{(3x1)}{\{\varepsilon\}} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$= \underset{(3x6)}{[B]} \; \underset{(6x1)}{\{d\}}$$

Thus the element stiffness matrix becomes,

$$\underset{(6x6)}{[K]} = \underset{(6x3)}{[B]^T} \; \underset{(3x3)}{[C]} \; \underset{(3x6)}{[B]} \, t\, A$$

TRIANGULAR ELEMENT

and the element nodal force vector due to thermal load is,

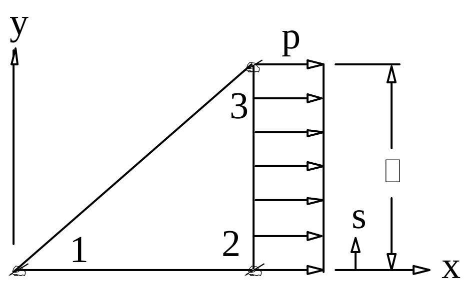
$$\underset{(6x1)}{\{F_0\}} = \underset{(6x3)}{[B]^T} \underset{(3x3)}{[C]} (T - T_0)\, t\, A\, \alpha \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

Similary, element load vector due to body forces is,

$$\{F_B\}^T = \frac{t\,A}{3} \{F_x \quad F_y \quad F_x \quad F_y \quad F_x \quad F_y\}$$

TRIANGULAR ELEMENT

For the traction, such as applied pressure along the edge between nodes 2-3,



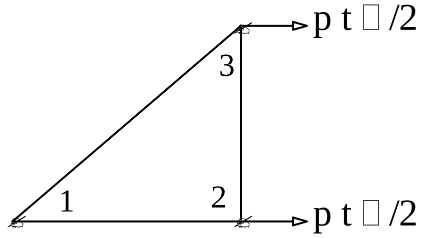
$$\{F_t\} = \begin{Bmatrix} F_{1x} & ? & 0 \\ F_{1y} & ? & 0 \\ F_{2x} & ? & 0 \\ F_{2y} & ? & 0 \\ F_{3x} & ? & 0 \\ F_{3y} & ? & 0 \end{Bmatrix}$$

TRIANGULAR ELEMENT

The non-zero nodal forces F_{2x} and F_{3x} are determined from,

$$\begin{Bmatrix} F_{2x} \\ F_{3x} \end{Bmatrix} = \int_0^{\ell} \begin{Bmatrix} 1 - \frac{s}{\ell} \\ \frac{s}{\ell} \end{Bmatrix} p\,(t\,ds) = p\,t\,\ell \begin{Bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{Bmatrix}$$

i.e., the applied pressure is represented by nodal forces,



$$\{F_t\} = \begin{Bmatrix} 0 \\ 0 \\ p\,t\,\ell/2 \\ 0 \\ p\,t\,\ell/2 \\ 0 \end{Bmatrix}$$

PLATE WITH CIRCULAR CUTOUT

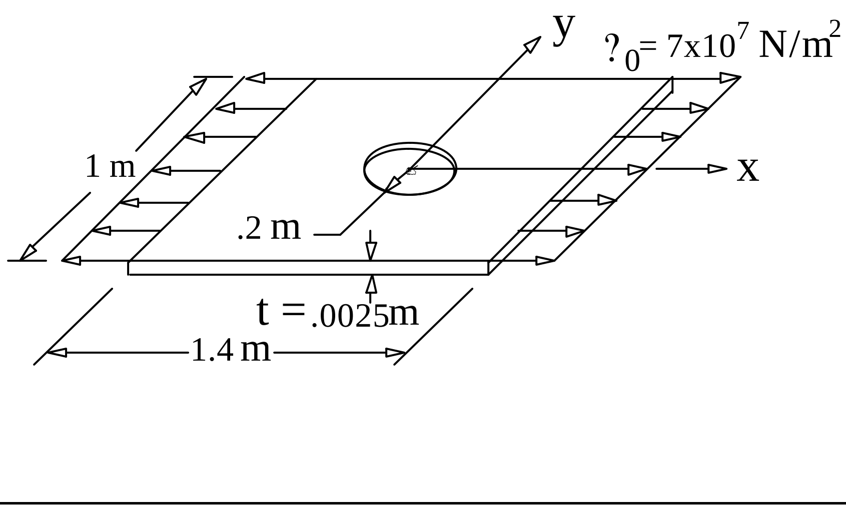


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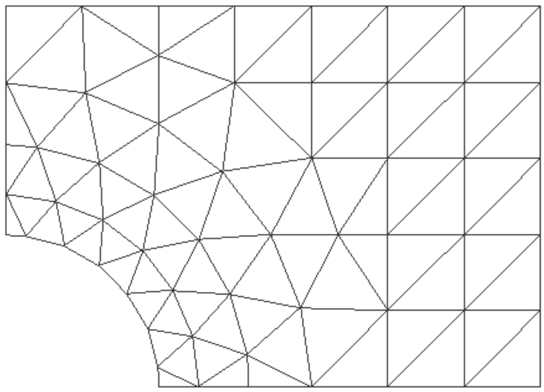
PLATE WITH CIRCULAR CUTOUT



FINITE

ELEMENT

MODEL

MODEL CONSISTS OF

58 NODES

85 TRIANGLES

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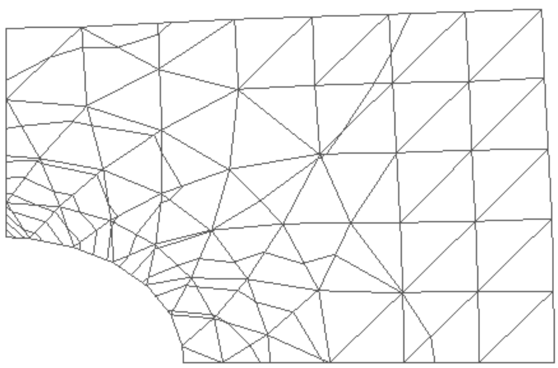
STRESS ?$_{x}$ ON DEFORMED SHAPE

SOLUTION CONTOURS FOR

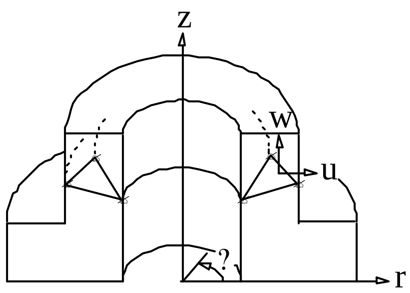
VARIABLE NUMBER 1



COLOR CONTOUR LEVELS

.200000E+08
.400000E+08
.600000E+08
.800000E+08
.100000E+09
.120000E+09
.140000E+09
.160000E+09
.180000E+09
.200000E+09
.220000E+09
.240000E+09
.260000E+09

AXISYMMETRIC PROBLEM



Equilibrium equations,

$$\frac{??_r}{?r} + \frac{??_{rz}}{?z} + \frac{?_r - ?_?}{r} + F_r = 0$$

$$\frac{??_{rz}}{?r} + \frac{??_z}{?z} + \frac{?_{rz}}{r} + F_z = 0$$

AXISYMMETRIC PROBLEM

Stress-strain relations,

$$\underset{(4x1)}{???} = \underset{(4x4)}{?C?}\underset{(4x1)}{??? - ?_0 ?}$$

where the stress and strain vectors are,

$$???^T = ??_r \quad ?_? \quad ?_z \quad ?_{rz} ?$$

$$???^T = ??_r \quad ?_? \quad ?_z \quad ?_{rz} ?$$

and the material elasticity matrix is,

$$?C? = \frac{E}{(1+?)(1-2?)} \begin{bmatrix} 1-? & ? & 0 & 0 \\ ? & 1-? & ? & 0 \\ 0 & ? & 1-? & 0 \\ 0 & 0 & 0 & \frac{1-2?}{2} \end{bmatrix}$$

AXISYMMETRIC ELEMENT

Assume element displacement distributions,

$$?? ? = \begin{matrix} u \\ w \end{matrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{matrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{matrix}$$

(2x1)

$$= ?N? \; ???$$

(2x6) (6x1)

where N_i, i = 1, 2, 3 are the element interpolation functions given by,

AXISYMMETRIC ELEMENT

$$N_i = \frac{1}{2A}(a_i + b_i r + c_i z) \qquad i = 1, 2, 3$$

Then the strain components can be determined from the strain-displacement relations,

$$??? = \begin{matrix} ?_r \\ ?_? \\ ?_z \\ ?_{rz} \end{matrix} = \begin{matrix} \frac{?u}{?r} \\ \frac{u}{r} \\ \frac{?w}{?z} \\ \frac{?u}{?z} ? \frac{?w}{?r} \end{matrix} = \frac{1}{2A} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ \frac{2AN_1}{r} & 0 & \frac{2AN_2}{r} & 0 & \frac{2AN_3}{r} & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \begin{matrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{matrix}$$

(4x1)

$$= ?B(r,z)\; ????$$

(4x6) (6x1)

AXISYMMETRIC ELEMENT

Knowing the strain-displacement matrix, $[B(r,z)]$, the element stiffness matrix can be derived from,

$$[K] = \int_V [B]^T [C] [B] dV$$

with $dV = 2\pi \, r \, dr \, dz$, to yield,

$$\underset{(6x6)}{[K]} = 2\pi \int_V \underset{(6x4)}{[B(r,z)]^T} \underset{(4x4)}{[C]} \underset{(4x6)}{[B(r,z)]} r \, dr \, dz$$

AXISYMMETRIC ELEMENT

Evaluation of this stiffness matrix may use:

1. Closed-form integration by employing algebraic manipulation programs, such as MACSYMA, MATHEMATICA, etc.

2. Numerical integration, e.g., Gauss- Legendre integration.

3. Closed-form integration by approximating r and z at the element centroid so that evaluation is much easier.

AXISYMMETRIC ELEMENT

Similary, element load vector due to temperature change is,

$$\underset{(6\times1)}{\{F_0\}} = 2\pi \int_A \underset{(6\times4)}{[B(r,z)]^T} \underset{(4\times4)}{[C]} \{\alpha\} (T(r,z) - T_0) \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix} r\,dr\,dz$$

and due to the body forces is,

$$\{F_B\} = 2\pi \int_A [B(r,z)]^T \begin{Bmatrix} F_r \\ F_z \end{Bmatrix} r\,dr\,dz$$

AXISYMMETRIC ELEMENT

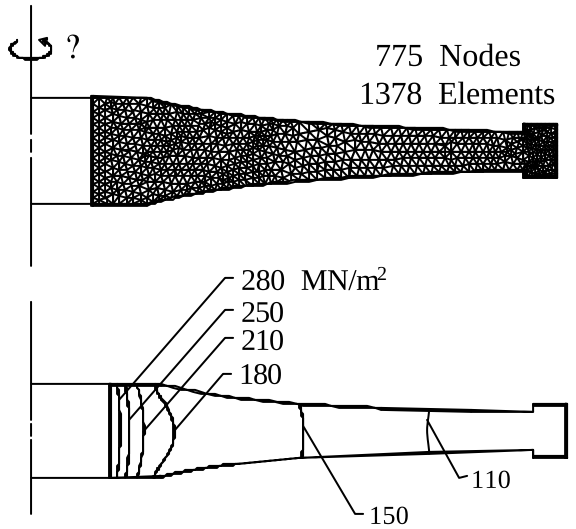
$$\{F_B\} = 2\pi \int_A [B(r,z)]^T \begin{Bmatrix} F_r \\ F_z \end{Bmatrix} r\,dr\,dz$$

where F_r and F_z are the body forces in r and z directions, respectively. For centrifugal force in radial r direction,

$$F_r = \rho\, r\, \omega^2$$

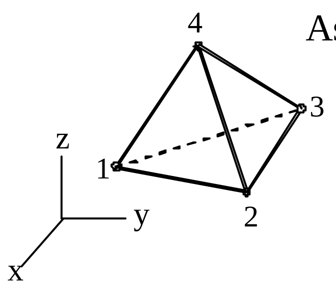
where ρ is the material density and ω is the angular velocity.

TURBINE ROTOR



THREE-DIMENSIONAL PROBLEM

Tetrahedral Element



Assume displacement distributions,

$$u(x, y, z) = ?N(x, y, z)??u? \quad (1x4) \quad (4x1)$$

$$v(x, y, z) = ?N(x, y, z)??v? \quad (1x4) \quad (4x1)$$

$$w(x, y, z) = ?N(x, y, z)??w? \quad (1x4) \quad (4x1)$$

where the element interpolation functions,

$$N_i(x, y, z) = \frac{1}{6V}(a_i + b_i x + c_i y + d_i z) \qquad i = 1, 2, 3, 4$$

TETRAHEDRAL ELEMENT

$$N_i = \frac{1}{6V}(a_i + b_i x + c_i y + d_i z) \qquad i = 1, 2, 3, 4$$

where

$$V = \text{Element volume} = \frac{1}{6}\begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix}$$

TETRAHEDRAL ELEMENT

$$a_1 = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} \qquad c_1 = -\begin{vmatrix} x_2 & 1 & z_2 \\ x_3 & 1 & z_3 \\ x_4 & 1 & z_4 \end{vmatrix}$$

$$b_1 = -\begin{vmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix} \qquad d_1 = -\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$$

Other constants a_i, b_i, c_i, d_i, $i = 2, 3, 4$ are determined in the same way using cyclic permutation.

TETRAHEDRAL ELEMENT

Then the strain-displacement relations are,

$$\{\varepsilon\}_{(6x1)} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{Bmatrix} = [B(x, y, z)]_{(6x12)} \{d\}_{(12x1)}$$

TETRAHEDRAL ELEMENT

where $[B(x, y, z)]$ is the strain-displacement matrix and,

$$\{d\}^T = \{u_1 \quad v_1 \quad w_1 \quad u_2 \quad v_2 \quad w_2 \ldots\ldots u_4 \quad v_4 \quad w_4\}$$

Thus the element stiffness matrix can be derived from,

$$[K] = \int_V [B]^T [C][B] dV$$

For tetrahedral element, the matrix $[B]$ is constant, then

TETRAHEDRAL ELEMENT

$$\underset{(12\times12)}{[K]} = \underset{(12\times6)}{[B]^T}\,\underset{(6\times6)}{[C]}\,\underset{(6\times12)}{[B]}\;V$$

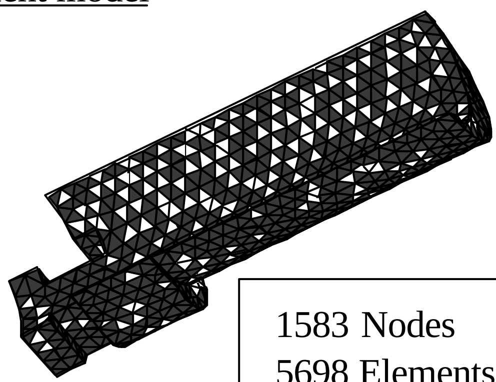
Similary, element load vector due to temperature change is,

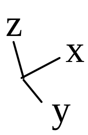
$$\underset{(12\times1)}{\{F_0\}} = \underset{(12\times6)}{[B]^T}\,\underset{(6\times6)}{[C]}\,?\,(T(x,y,z) - T_0)\,V \begin{Bmatrix}1\\1\\1\\0\\0\\0\end{Bmatrix}$$

Other load vectors can be evaluated in the same fashion.

STEAM TURBINE BLADE

Finite element model



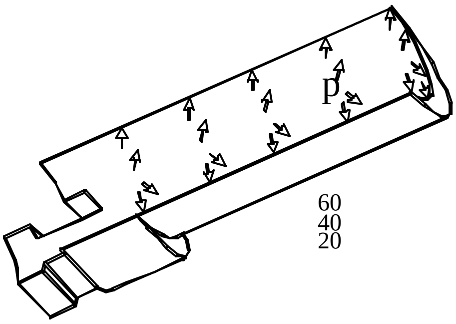


1583 Nodes
5698 Elements

336

STEAM TURBINE BLADE

Applied Pressure



Von Mises Stress

40 MN/m²
20
10

60
40
20

10
20
40
60

60
40
20

337

STEAM TURBINE BLADE

Centrifugal Force

Von Mises Stress

70 MN/m²
30
20
10

20

180
100
70

STEAM TURBINE BLADE

Temperature Profile

Von Mises Stress

T_2

15 MN/m²

10

5

T_1

10

15

10

STEAM TURBINE BLADE

Von Mises Stress

30 MN/m²

60

90
110
200
250

60
110
200

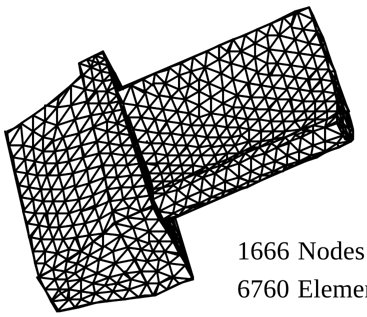
Combined loads:

- Pressure load
- Centrifugal force
- Thermal load

340

GAS TURBINE BLADE

F. E Model



1666 Nodes
6760 Elements

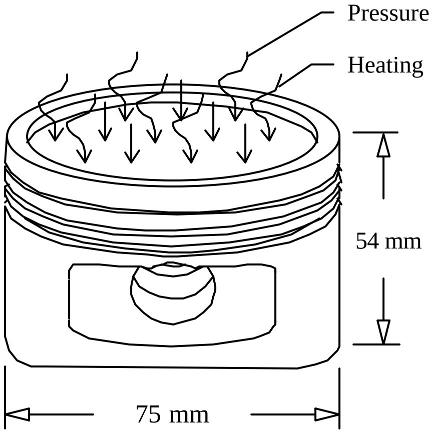
Von Mises Stress form
Combined Loads

100 MN/m^2
150

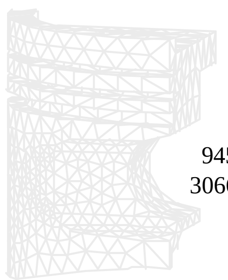
100
200
250

341

MOTORCYCLE PISTON

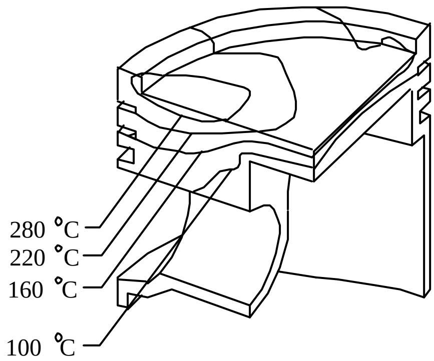


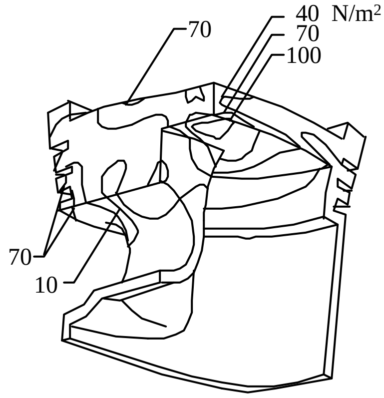
F. E. Model



945 Nodes
3066 Elements

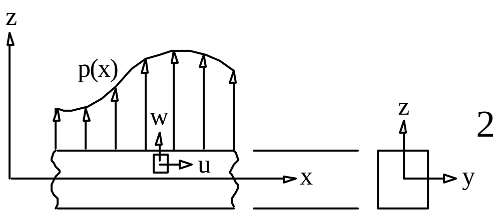
MOTORCYCLE PISTON





BEAM BENDING

Basic assumptions:



1. Plane section remains plane,

$$u = -z\frac{dw}{dx}$$

2. Long and shedder beam,

$$w = w(x)$$

These assumptions lead to the governing differential eq.,

$$\frac{d^2}{dx^2}\left(E\,I\,\frac{d^2w}{dx^2}\right) - p(x) = 0$$

where E is the modulus of elasticity and I is the moment of inertia of area.

BEAM ELEMENT

Finite element equations can be derived easily using the variational approach. Since the internal strain energy is,

$$U^{?} = \frac{1}{2}\int_V \sigma_x \varepsilon_x \, dV$$

Substitute $\sigma_x = E\varepsilon_x$ where $\varepsilon_x = \frac{du}{dx} = -z\frac{d^2w}{dx^2}$ from the first assumption,

$$U^{?} = \frac{1}{2}\int_V E z^2 \left(\frac{d^2w}{dx^2}\right)^2 dV$$

For beam of length L and cross-sectional area A, then

BEAM ELEMENT

$$U^{?} = \frac{1}{2}\int_0^L \left[\int_A E z^2 \left(\frac{d^2w}{dx^2}\right)^2 dA\right] dx$$

Using the second assumption, w = w(x), then

$$U^{?} = \frac{1}{2}\int_0^L E \left[\int_A z^2 \, dA\right] \left(\frac{d^2w}{dx^2}\right)^2 dx$$

[illegible]
I

where I is the moment of inertia of area, e.g. $I = bh^3/12$ for rectangular beam section of width b and height h.

BEAM ELEMENT

Or $$U^{?} = \frac{1}{2}\int_0^L EI\left(\frac{d^2w}{dx^2}\right)^2 dx$$

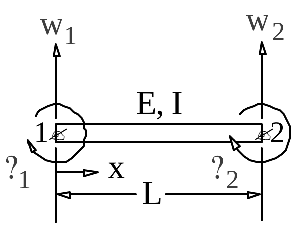
The potential energy from external force is,

$$V^{?} = -\,\text{work} = -\int_0^L p(x)\,w(x)\,dx$$

Thus, the total potential energy is,

$$J = \frac{1}{2}\int_0^L EI\left(\frac{d^2w}{dx^2}\right)^2 dx - \int_0^L p(x)\,w(x)\,dx$$

BEAM ELEMENT



Assume deflection distributions,

$$w(x) = ?_1 + ?_2 x + ?_3 x^2 + ?_4 x^3$$

where $?_i$, i = 1, 2, 3, 4 are constants to be determined from the conditions at nodes,

$$w(x = 0) = w_1 \qquad w(x = L) = w_2$$

$$-\frac{dw}{dx}(x ? 0) = ?_1 \qquad -\frac{dw}{dx}(x ? L) = ?_2$$

BEAM ELEMENT

These lead to the element deflection in the form,

$$w(x) \;=\; \lfloor N_1 \;\; N_2 \;\; N_3 \;\; N_4 \rfloor \begin{Bmatrix} w_1 \\ ?_1 \\ w_2 \\ ?_2 \end{Bmatrix} \;=\; \underset{(1\text{x}4)}{\lfloor N \rfloor} \underset{(4\text{x}1)}{\{?\}}$$

where $N_1 = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$ $\quad N_3 = \left(\frac{x}{L}\right)^2 \left(3 - 2\frac{x}{L}\right)$

$$N_2 = -x\left(\frac{x}{L} - 1\right)^2 \qquad N_4 = -\frac{x^2}{L}\left(\frac{x}{L} - 1\right)$$

BEAM ELEMENT

To derive the element eqs., the total potential energy is first written in the form,

$$J = \frac{1}{2}\int_0^L \left(\frac{d^2w}{dx^2}\right) EI \left(\frac{d^2w}{dx^2}\right) dx \;-\; \int_0^L p(x)\; w(x)\; dx$$

Since, $\quad w(x) = \underset{(1\text{x}4)}{\lfloor N \rfloor} \underset{(4\text{x}1)}{\{?\}} = \underset{(1\text{x}4)}{\lfloor ? \rfloor} \underset{(4\text{x}1)}{\{N\}}$

then, $$\frac{d^2w}{dx^2} = \underset{(1\text{x}4)}{\left\lfloor \frac{d^2N}{dx^2} \right\rfloor} \underset{(4\text{x}1)}{\{?\}} = \underset{(1\text{x}4)}{\lfloor ? \rfloor} \underset{(4\text{x}1)}{\left\{ \frac{d^2N}{dx^2} \right\}}$$

BEAM ELEMENT

Then,

$$J = \frac{1}{2}\int_0^L \{\delta\}^T \left[\frac{d^2N}{dx^2}\right]^T EI \left[\frac{d^2N}{dx^2}\right] \{\delta\}\, dx - \int_0^L p(x)\, w(x)\, dx$$

$$= \frac{1}{2} \underset{(1x4)}{\{\delta\}^T} \underbrace{\int_0^L \left[\frac{d^2N}{dx^2}\right]^T EI \left[\frac{d^2N}{dx^2}\right] dx}_{\underset{(4x4)}{[K]}} \underset{(4x1)}{\{\delta\}}$$

$$- \underset{(1x4)}{\{\delta\}^T} \underbrace{\int_0^L p(x) \{N\}\, dx}_{\underset{(4x1)}{\{F\}}}$$

BEAM ELEMENT

Or,

$$J = \frac{1}{2}\{\delta\}^T [K] \{\delta\} - \{\delta\}^T \{F\}$$

To derive the element equations, the total PE. statement is minimized,

$$\frac{\partial J}{\partial \{\delta\}} = 0$$

to yield,

$$\underset{(4x4)}{[K]}\ \underset{(4x1)}{\{\delta\}} = \underset{(4x1)}{\{F\}}$$

BEAM ELEMENT EQUATIONS

$$[K]\{\Delta\} = \{F\}$$
$$(4\text{x}4)\ (4\text{x}1) \qquad (4\text{x}1)$$

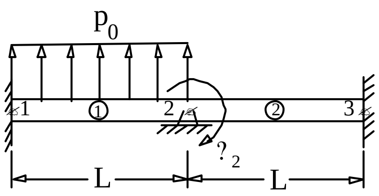
where the element stiffness matrix,

$$[K] = \int_0^L \left[\frac{d^2N}{dx^2}\right]^T EI \left[\frac{d^2N}{dx^2}\right] dx = \frac{2EI}{L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ & 2L^2 & 3L & L^2 \\ & & 6 & 3L \\ \text{Sym} & & & 2L^2 \end{bmatrix}$$

and the element load vector due to uniform load $p(x) = p_0$,

$$\{F\} = \int_0^L p(x)\{N\}dx = \begin{Bmatrix} p_0 L/2 \\ -p_0 L^2/12 \\ p_0 L/2 \\ p_0 L^2/12 \end{Bmatrix}$$

BEAM ELEMENT



Example Compute the slope at node 2 of the beam below,

Element equations for element ① are;

$$\frac{2EI}{L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ & 2L^2 & 3L & L^2 \\ & & 6 & 3L \\ \text{Sym} & & & 2L^2 \end{bmatrix}\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} p_0 L/2 \\ -p_0 L^2/12 \\ p_0 L/2 \\ p_0 L^2/12 \end{Bmatrix}$$

Element equations for element ② are in the same form except $p_0 = 0$.

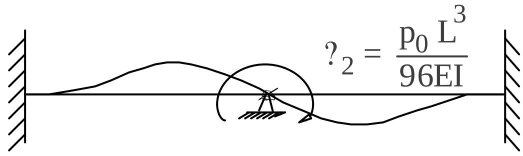
BEAM ELEMENT

Assembling element equations and apply BC's to yield,

$$\frac{2EI}{L^3}\begin{bmatrix} 6 & -3L & -6 & -3L & 0 & 0 \\ & 2L^2 & 3L & L^2 & 0 & 0 \\ & & 12 & 0 & -6 & -3L \\ & & & 4L^2 & -3L & L^2 \\ & & & & 6 & 3L \\ Sym & & & & & 2L^2 \end{bmatrix}\begin{Bmatrix} w_1 \\ ?_1 \\ w_2 \\ ?_2 \\ w_3 \\ ?_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ ? \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} p_0 L/2 \\ -p_0 L^2/12 \\ p_0 L/2 \\ p_0 L^2/12 \\ 0 \\ 0 \end{Bmatrix}$$

then solve to get,

$$?_2 = \frac{p_0 L^3}{96EI}$$



STRUCTURAL DYNAMICS

Recall FE eqs. for static problem,

$$\underset{(24x24)}{?K?}\ \underset{(24x1)}{???} = \underset{(24x1)}{?F_0?} + \underset{(24x1)}{?F_B?} + \underset{(24x1)}{?F_t?}$$

where the matrix sizes correspond to 8-node hexahedral element. The load vector due to the body forces is,

$$\underset{(24x1)}{?F_B?} = \int_V \underset{(24x3)}{?N?^T}\ \underset{(3x1)}{?F?}\, dV$$

where $\underset{(3x1)}{?F?}$ consists of body forces in the x, y and z directions.

For structural dynamics, the inertia force (varies with acceleration) and the damping force (varies with velocity) must be included. These can be done easily through the body force according to the D'Alembert principle, i.e.,

STRUCTURAL DYNAMICS

Inertia force,

$$\{F\} = -\rho\{\ddot{u}\} = -\rho\begin{Bmatrix}\ddot{u}\\ \ddot{v}\\ \ddot{w}\end{Bmatrix}$$

Damping force,

$$\{F\} = -c\{\dot{u}\} = -c\begin{Bmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\end{Bmatrix}$$

Thus, the new equivalent body force vector is,

$$\underset{(24\text{x}1)}{\{F_B\}} = \int_V \underset{(24\text{x}3)}{[N]^T}\left(\underset{(3\text{x}1)}{\{F\}} - \rho\underset{(3\text{x}1)}{\{\ddot{u}\}} - c\underset{(3\text{x}1)}{\{\dot{u}\}}\right) dV$$

Then, the FE eqs. for dynamic problem are,

STRUCTURAL DYNAMICS

$$\underset{(24\text{x}24)}{[M]}\underset{(24\text{x}1)}{\{\ddot{q}\}} + \underset{(24\text{x}24)}{[C]}\underset{(24\text{x}1)}{\{\dot{q}\}} + \underset{(24\text{x}24)}{[K]}\underset{(24\text{x}1)}{\{q\}} = \underset{(24\text{x}1)}{\{F_0\}} + \underset{(24\text{x}1)}{\{F_B\}} + \underset{(24\text{x}1)}{\{F_t\}}$$

Here, $[M]$ is the mass matrix,

$$\underset{(24\text{x}24)}{[M]} = \int_V \rho\,\underset{(24\text{x}3)}{[N]^T}\,\underset{(3\text{x}24)}{[N]}\,dV$$

where ρ is the material density, and $[C]$ is the damping matrix,

$$\underset{(24\text{x}24)}{[C]} = \int_V c\,\underset{(24\text{x}3)}{[N]^T}\,\underset{(3\text{x}24)}{[N]}\,dV$$

where c is the damping coefficient. These two matrices can be evaluated in closed-form or by numerical integration.

MASS MATRIX

As an example, the mass matrix for 2-node truss element is,

$$[M] = \int_0^L \rho [N]^T [N] (A\,dx) = \rho A \int_0^L \begin{Bmatrix} 1-\dfrac{x}{L} \\ \dfrac{x}{L} \end{Bmatrix} \begin{bmatrix} 1-\dfrac{x}{L} & \dfrac{x}{L} \end{bmatrix} dx$$

$$[M] = \rho AL \begin{bmatrix} \dfrac{1}{3} & \dfrac{1}{6} \\ \dfrac{1}{6} & \dfrac{1}{3} \end{bmatrix}$$

- Consistent mass matrix
- Solve coupled equations

If coefficients are combined and placed at diagonal terms,

$$[M] = \rho AL \begin{bmatrix} \dfrac{1}{2} & 0 \\ 0 & \dfrac{1}{2} \end{bmatrix}$$

- Lumped mass matrix
- Solve uncoupled equations

STRUCTURAL DYNAMICS

In conclusion, FE eqs. for dynamic problem are,

$$[M]\{\ddot{D}\} + [C]\{\dot{D}\} + [K]\{D\} = \{F(t)\}$$

where $\{F(t)\}$ is the forcing function that includes the applied external force, body force, thermal load, etc.

<u>Solution methods widely used for system eqs.</u>:

1. Modal Superposition
2. Recurrence Formula

360

MODAL SUPERPOSITION

The approach consists of 2 steps:

1. Finding eigenvalues and eigenvectors
2. Solving for dynamic response

<u>Step 1</u>: To find eigenvalues and eigenvectors, need to do free vibration, i.e., solve

$$[M]\{\ddot{x}\} + [K]\{x\} = 0$$

Assume $\quad \{x\} = \{\phi\}\sin\omega t$

then $\quad \{\ddot{x}\} = -\omega^2\{\phi\}\sin\omega t$

where ω denotes the frequencies and $\{\phi\}$ represents the mode shapes. Substitute in the above eq. to get,

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MODAL SUPERPOSITION

$$\left([K] - \omega^2[M]\right)\{\phi\} = \{0\}$$

which means the determinant must be zero,

$$\left|[K] - \omega^2[M]\right| = 0$$

This leads to a polynomial eq. in the form,

$$\left(\omega^2\right)^n + D_1\left(\omega^2\right)^{n-1} + \ldots\ldots\ldots + D_{n-1}\left(\omega^2\right) + D_n = 0$$

that can be solved for frequencies, ω_i, and the corresponding mode shapes, $\{\phi\}_i$. In this eq., D_i, i = 1, n are constants and n is the total number of eqs.

MODAL SUPERPOSITION

Note that the mode shapes have orthogonal property, i.e.,

$$\{\phi\}_i^T [K] \{\phi\}_j = 0 \qquad i \neq j$$

$$\{\phi\}_i^T [M] \{\phi\}_j = 0 \qquad i \neq j$$

But when i = j, we have

$$\{\phi\}_i^T [K] \{\phi\}_i = K_{ii}^{*}$$

$$\{\phi\}_i^T [M] \{\phi\}_i = M_{ii}^{*}$$

MODAL SUPERPOSITION

<u>Step 2</u>: To solve for dynamic response, we write the nodal response as,

$$\{U\} = \underbrace{\left[\underset{(nx1)}{\{\phi\}_1} \; \underset{(nx1)}{\{\phi\}_2} \; \underset{(nx1)}{\{\phi\}_3} \; \cdots \; \underset{(nx1)}{\{\phi\}_n} \right]}_{\underset{(nxn)}{[A]}} \{\eta(t)\}$$

where $[A]$ is the square matrix that includes all mode shapes and $\{\eta(t)\}$ is the vector of modal amplitudes which will be determined. Substitute this eq. into the element eqs. and premultiply by $[A]^T$ to get,

MODAL SUPERPOSITION

$$\underbrace{[A]^T[M][A]}_{[M^*]}\{\ddot{q}\} + \underbrace{[A]^T[C][A]}_{[C^*]}\{\dot{q}\} + \underbrace{[A]^T[K][A]}_{[K^*]}\{q\} = \underbrace{[A]^T\{F(t)\}}_{\{F^*\}}$$

Or, $[M^*]\{\ddot{q}\} + [C^*]\{\dot{q}\} + [K^*]\{q\} = \{F^*\}$

From the orthogonal property, these $[M^*]$, $[C^*]$ and $[K^*]$ matrices are called generalized matrices and are all diagonal matrices, thus the equations are uncoupled.

MODAL SUPERPOSITION

Since, $[K]\{\phi\}_j = \omega_j^2 [M]\{\phi\}_j$

Premultiply by $\{\phi\}_i^T$ and use orthogonal property (not summed on i), $K^*_{ii} = \omega_i^2 M^*_{ii}$

In addition, the diagonal terms in the generalized $[C^*]$ matrix may be written in form of the generalized mass matrix, $[C^*_{ii}] = 2\,\zeta_i\,\omega_i\,M^*_{ii}$

where ζ_i is called the damping ratio. Then the element equations of motion become,

MODAL SUPERPOSITION

$$M^{?}_{ii}\, ?_{i} + 2\,?_{i}\, ?_{i}\, M^{?}_{ii}\, ?_{i} + ?^{2} M^{?}_{ii}\, ?_{i} = F^{?}_{i}$$

Or,

$$?_{i} + 2\,?_{i}\, ?_{i}\, ?_{i} + ?^{2}\, ?_{i} = \frac{F^{?}_{i}}{M^{?}_{ii}} \qquad i = 1, 2, ..., n$$

Once the modal amplitudes, $?_{i}(t)$ are solved, the nodal response are determined from,

$$??? = ?A??? (t)?$$

RECURRENCE RELATIONS

Will consider two popular methods:

1. Central Difference method

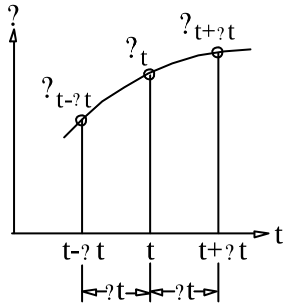
   This leads to "Explicit algorithm", i.e., solve uncoupled equations.

2. Newmark Method

   This leads to "Implicit algorithm", i.e., solve simultaneous coupled equations but avoid numerical instability.

CENTRAL DIFFERENCE METHOD

Consider transient response $?_{t-?t}, ?_t$ and $?_{t+?t}$ of a typical node at time t-? t, t and t+? t, receptively, where ? t is the “time step”. Apply Taylor series expansion on the response at time t-? t and t+? t,



$$?_{t+?t} \;=\; ?_t + ?_t(?t) + \frac{1}{2!}?_t(?t)^2 + O(?t)^3$$

$$?_{t-?t} \;=\; ?_t - ?_t(?t) + \frac{1}{2!}?_t(?t)^2 - O(?t)^3$$

CENTRAL DIFFERENCE METHOD

Subtract to get,

$$?_{t??t} - ?_{t-?t} \;=\; 2?(?t) + O(?t)^3$$

then omit higher order term to give,

$$?_t \;?\; \frac{?_{t+?t} - ?_{t-?t}}{2?t}$$

Similary, add the two eqs. together to give,

$$?_{t??t} + ?_{t-?t} \;=\; 2?_t + ?_t(?t)^2 + O(?t)^4$$

and omit higher order term to yield,

$$?_t \;?\; \frac{?_{t+?t} - 2?_t + ?_{t-?t}}{(?t)^2}$$

CENTRAL DIFFERENCE METHOD

Then, from the equation of motion at time t,

$$[M]\{\ddot{\Delta}\}_t + [C]\{\dot{\Delta}\}_t + [K]\{\Delta\}_t = \{F(t)\}$$

Substitute the first and second derivative approximations, and rearrange terms to give,

$$\left[\frac{1}{(\Delta t)^2}[M] + \frac{1}{2\Delta t}[C]\right]\{\Delta\}_{t+\Delta t} = \{F(t)\}$$

$$-\left[[K] - \frac{2}{(\Delta t)^2}[M]\right]\{\Delta\}_t - \left[\frac{1}{(\Delta t)^2}[M] - \frac{1}{2\Delta t}[C]\right]\{\Delta\}_{t-\Delta t}$$

CENTRAL DIFFERENCE METHOD

or, in short, $[\bar{K}]\{\Delta\}_{t+\Delta t} = \{\bar{F}\}$

where $[\bar{K}] = \frac{1}{(\Delta t)^2}[M] + \frac{1}{2\Delta t}[C]$

$$\{\bar{F}\} = \{F(t)\} - \left[[K] - \frac{2}{(\Delta t)^2}[M]\right]\{\Delta\}_t$$

$$- \left[\frac{1}{(\Delta t)^2}[M] - \frac{1}{2\Delta t}[C]\right]\{\Delta\}_{t-\Delta t}$$

CENTRAL DIFFERENCE METHOD

Note that $\left[\hat{K}\right]\{u\}_{t+\Delta t} = \left\{\hat{F}\right\}$

where $\left[\hat{K}\right] = \frac{1}{(\Delta t)^2}[M] + \frac{1}{2\Delta t}[C]$

The above are coupled eqs. e.g., $[M]_{truss} = \rho AL \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}$.

If both [M] and [C] are lumped, eg., $[M]_{lumped} = \rho AL \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$,

then the above eqs. become uncoupled which can be solved explicitly and is called “Explicit algorithm”.

CENTRAL DIFFERENCE METHOD

$$\left[\hat{K}\right]_{lumped}\{u\}_{t+\Delta t} = \left\{\hat{F}\right\}$$

Also note that stable solution is obtained if,

$$\Delta t \le \Delta t_{cr} = \frac{2}{\omega_{max}}$$

where Δt_{cr} is the critical time step and ω_{max} is the maximum frequency (shown by example later).

At starting point, t=0, know $\{u\}_{t=0}$ and $\{\dot{u}\}_{t=0}$ and also need $\{u\}_{-\Delta t}$ which can be determined from first computing $\{\ddot{u}\}_{t=0}$ as follows,

CENTRAL DIFFERENCE METHOD

$$[M]\{\ddot{u}\}_{t=0} = \{F(t=0)\} - [C]\{\dot{u}\}_{t=0} - [K]\{u\}_{t=0}$$

then use Taylor series at $-\Delta t$, i.e.,

$$\{u\}_{-\Delta t} = \{u\}_{t=0} - \{\dot{u}\}_{t=0}\Delta t + \frac{1}{2}\{\ddot{u}\}_{t=0}(\Delta t)^2$$

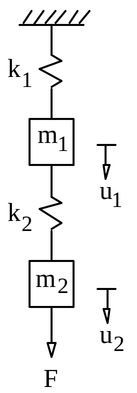
Example Given the equations of motion of a mass-spring system,

$$\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}}_{[M]} \underbrace{\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix}}_{\{\ddot{u}\}} + \underbrace{\begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}}_{[K]} \underbrace{\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}_{\{u\}} = \underbrace{\begin{Bmatrix} 0 \\ 10 \end{Bmatrix}}_{\{F\}}$$

CENTRAL DIFFERENCE METHOD

with initial conditions of:

$$\{u\}_{t=0} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_{t=0} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \text{ and } \{\dot{u}\}_{t=0} = \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix}_{t=0} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$



Determine the transient response for $0 \le t \le 2.8$ using the time step $\Delta t = 0.28$

Note that the critical time step can be determined by the maximum frequency from the eigenvalue problem,

CENTRAL DIFFERENCE METHOD

$$\left| ?K? - ?^2 ?M? \right| = 0$$

Substitute
$$\begin{vmatrix} 6 - 2?^2 & -2 \\ -2 & 4 - ?^2 \end{vmatrix} = 0$$

$$(6 - 2?^2)(4 - ?^2) - (-2)(-2) = 0$$

$$?^4 - 7?^2 + 10 = 0$$

$$(?^2 - 2)(?^2 - 5) = 0$$

$$?_1^2, \; ?_2^2 = 2, \; 5$$

$$?_1 = \sqrt{2} = 1.414 \quad \text{and} \quad ?_2 = \sqrt{5} = 2.236$$

i.e., $?_{max} = 2.236$

CENTRAL DIFFERENCE METHOD

Thus, the critical time step is,

$$?t_{cr} = \frac{2}{?_{max}} = \frac{2}{2.236} = 0.89$$

This means, the given time step of ?t=0.28 won't produce unstable (diverged) solution.

At the starting point, t=0, need to know displacements at ?t = -0.28 which can be computed from,

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix}_{t=0} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} - \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_{t=0} ? \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

CENTRAL DIFFERENCE METHOD

to give $\{\ddot{U}\}_{t=0} = \begin{Bmatrix}\ddot{u}_1\\ \ddot{u}_2\end{Bmatrix}_{t=0} = \begin{Bmatrix}0\\ 10\end{Bmatrix}$

Then, from Taylor series,

$$\{U\}_{-\Delta t} = \{U\}_{-0.28} = \begin{Bmatrix}0\\ 0\end{Bmatrix} - \begin{Bmatrix}0\\ 0\end{Bmatrix}(0.28) + \frac{1}{2}\begin{Bmatrix}0\\ 10\end{Bmatrix}(0.28)^2$$

thus,

$$\{U\}_{-\Delta t} = \{U\}_{-0.28} = \begin{Bmatrix}u_1\\ u_2\end{Bmatrix}_{0.28} = \begin{Bmatrix}0\\ 0.392\end{Bmatrix}$$

Then, solutions at different time steps can be computed as follows:

CENTRAL DIFFERENCE METHOD

<u>First time step</u>: (t = 0, t - Δt = -0.28, t + Δt = 0.28)

$$\frac{1}{(0.28)^2}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}_{0.28} = \begin{Bmatrix}0\\ 10\end{Bmatrix} - \left(\begin{bmatrix}6 & -2\\ -2 & 4\end{bmatrix} - \frac{2}{(0.28)^2}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\right)\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}_0$$

$$- \frac{1}{(0.28)^2}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}_{-0.28}$$

$$\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}_{0.28} = \begin{Bmatrix}0\\ 0.392\end{Bmatrix}$$

CENTRAL DIFFERENCE METHOD

Second time step: (t = 0.28, t - ? t = 0, t + ? t = 0.56)

$$\frac{1}{(0.28)^2}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}_{0.56} = \begin{Bmatrix}0\\ 10\end{Bmatrix} - \left(\begin{bmatrix}6 & -2\\ -2 & 4\end{bmatrix} - \frac{2}{(0.28)^2}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\right)\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}_{0.28}$$

$$- \frac{1}{(0.28)^2}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}_{0}$$

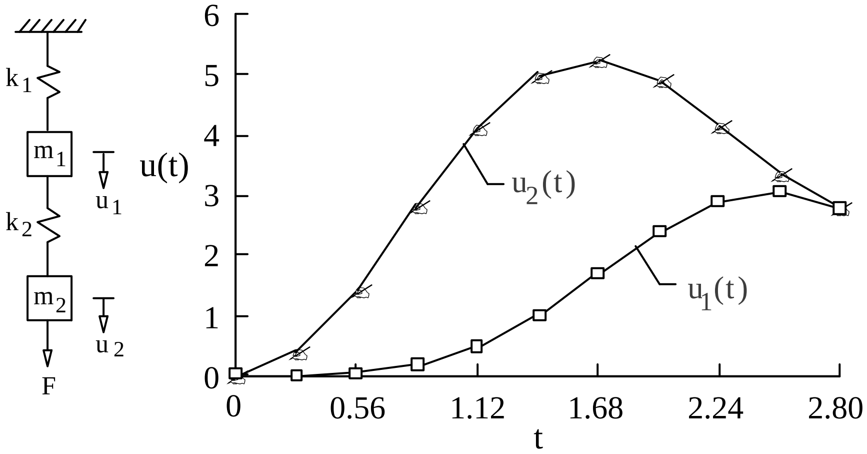
$$\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}_{0.56} = \begin{Bmatrix}0.0307\\ 1.445\end{Bmatrix}$$

and so on.

RESULT OF TRANSIENT RESPONSE

Time step	time, t	$u_1(t)$	$u_2(t)$
	0.	0.	0.
1	0.28	0.	0.39
2	0.56	0.03	1.45
3	0.84	0.17	2.83
4	1.12	0.49	4.14
5	1.40	1.02	5.02
6	1.68	1.70	5.26
7	1.96	2.40	4.90
8	2.24	2.91	4.17
9	2.52	3.07	3.37
10	2.80	2.77	2.78

RESULT OF TRANSIENT RESPONSE



NEWMARK METHOD

<u>Key idea</u> Use "Average Acceleration" over time step between time t and $t+\Delta t$,

$$\ddot{u}_{avg} = \frac{\ddot{u}_t + \ddot{u}_{t+\Delta t}}{2}$$

then write nodal displacement and velocity at time $t+\Delta t$ in the form,

$$u_{t+\Delta t} = u_t + \dot{u}_t \Delta t + \frac{1}{2}\ddot{u}_{avg} (\Delta t)^2$$

$$\dot{u}_{t+\Delta t} = \dot{u}_t + \ddot{u}_{avg} \Delta t$$

NEWMARK METHOD

Substitute $\{\ddot{\Delta}\}_{avg}$ into these 2 eqs. and rearrange term to get,

$$\{\ddot{\Delta}\}_{t+\Delta t} = \frac{4}{(\Delta t)^2}\{\Delta\}_{t+\Delta t} - \frac{4}{(\Delta t)^2}\{\Delta\}_t - \frac{4}{\Delta t}\{\dot{\Delta}\}_t - \{\ddot{\Delta}\}_t$$

and

$$\{\dot{\Delta}\}_{t+\Delta t} = \frac{2}{\Delta t}\{\Delta\}_{t+\Delta t} - \frac{2}{\Delta t}\{\Delta\}_t - \{\dot{\Delta}\}_t$$

which will be used in the equations of motion,

$$[M]\{\ddot{\Delta}\}_{t+\Delta t} + [C]\{\dot{\Delta}\}_{t+\Delta t} + [K]\{\Delta\}_{t+\Delta t} = \{F(t+\Delta t)\}$$

NEWMARK METHOD

Or, in short, $$[\bar{K}]\{\Delta\}_{t+\Delta t} = \{\bar{F}\}$$

where

$$[\bar{K}] = \frac{4}{(\Delta t)^2}[M] + \frac{2}{\Delta t}[C] + [K]$$

$$\{\bar{F}\} = \{F(t+\Delta t)\} + [C]\left(\frac{2}{\Delta t}\{\Delta\}_t + \{\dot{\Delta}\}_t\right) + [M]\left(\frac{4}{(\Delta t)^2}\{\Delta\}_t + \frac{4}{(\Delta t)}\{\dot{\Delta}\}_t + \{\ddot{\Delta}\}_t\right)$$

<u>Note</u>: Because these coupled eqs. must be solved, the method is called "<u>Implicit Algorithm</u>".

NEWMARK METHOD

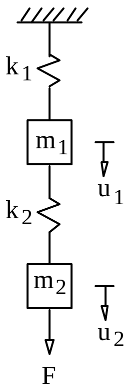
<u>Example</u> Given the equations of motion of a mass-spring system,

$$\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}}_{[M]} \underbrace{\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix}}_{\{\ddot{u}\}} + \underbrace{\begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}}_{[K]} \underbrace{\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}_{\{u\}} = \underbrace{\begin{Bmatrix} 0 \\ 10 \end{Bmatrix}}_{\{F\}}$$

with initial conditions of:

$$\{u\}_{t=0} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_{t=0} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \text{ and } \{\dot{u}\}_{t=0} = \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix}_{t=0} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

NEWMARK METHOD



Determine the transient response for $0 \le t \le 2.8$ using the time step $\Delta t = 0.28$

Note that the acceleration at t=0 can be determined from the above eqs.,

$$\{\ddot{u}\}_{t=0} = \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix}_{t=0} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

NEWMARK METHOD

First time step: (t = 0, t + Δt = 0.28)

$$\left[\frac{4}{(\Delta t)^2}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}6 & -2\\ -2 & 4\end{bmatrix}\right]\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}_{0.28}$$

$$= \begin{Bmatrix}0\\ 10\end{Bmatrix} + \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\left(\frac{4}{(0.28)^2}\begin{Bmatrix}0\\ 0\end{Bmatrix} + \frac{4}{(0.28)}\begin{Bmatrix}0\\ 0\end{Bmatrix} + \begin{Bmatrix}0\\ 10\end{Bmatrix}\right)$$

Or,

$$\begin{bmatrix}108.04 & -2\\ -2 & 55.02\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}_{0.28} = \begin{Bmatrix}0\\ 20\end{Bmatrix}$$

Solve these coupled eqs. to get,

$$\mathbf{U}_{t+\Delta t} = \begin{Bmatrix}u_1\\ u_2\end{Bmatrix}_{0.28} = \begin{Bmatrix}0.00673\\ 0.36379\end{Bmatrix}$$

TRANSIENT RESPONSE FROM NEWMARK METHOD

Time step	time, t	$u_1(t)$	$u_2(t)$
	0.	0.	0.
1	0.28	0.01	0.36
2	0.56	0.05	1.35
3	0.84	0.19	2.68
4	1.12	0.49	4.00
5	1.40	0.96	4.95
6	1.68	1.58	5.34
7	1.96	2.23	5.13
8	2.24	2.76	4.48
9	2.52	3.00	3.64
10	2.80	2.85	2.90

RESULT OF TRANSIENT RESPONSE