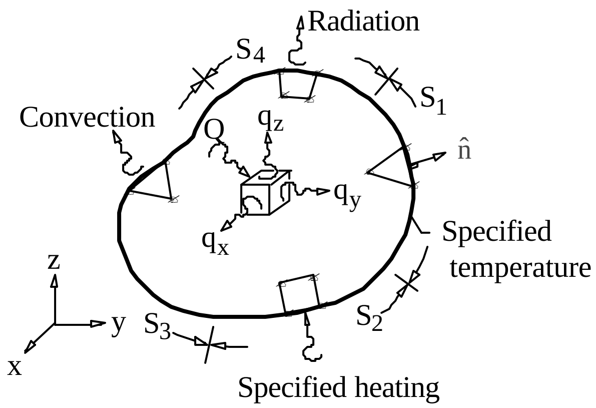
FINITE ELEMENT METHOD FOR HEAT TRANSFER PROBLEMS

HEAT TRANSFER PROBLEM



Governing differential equation,

$$-\left\{ \frac{?q_x}{?x} + \frac{?q_y}{?y} + \frac{?q_z}{?z} \right\} + Q = ?\,c\,\frac{?T}{?t}$$

HEAT TRANSFER PROBLEM

Heat flow rates can be written in terms of temperature gradients by Fourier's law,

$$\begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} = - \underbrace{\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}}_{[k]} \begin{Bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{Bmatrix}$$

where $[k]$ is the thermal conductivity matrix. As an example, for isotropic material,

$$[k] = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

HEAT TRANSFER PROBLEM

Boundary Conditions

(1) Specified temperature on S_1 :

$$T_S = T_1(x, y, z, t)$$

(2) Specified heating on S_2 :

$$q_x n_x + q_y n_y + q_z n_z = -q_S$$

(3) Convection heat transfer on S_3 :

$$q_x n_x + q_y n_y + q_z n_z = h(T_S - T_\infty)$$

HEAT TRANSFER PROBLEM

Boundary Conditions (Cont.)

(4) Radiation heat transfer on S_4 :

$$q_x n_x + q_y n_y + q_z n_z = ? ? \left(T_S^4 - \overline{T}_?^4 \right) - ? q_r$$

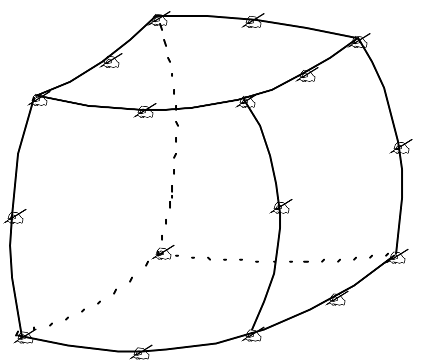
Initial Condition

$$T(x, y, z, 0) = T_0(x, y, z)$$

FINITE ELEMENT EQUATIONS

Assume element temperature distribution,

$$T(x, y, z, t) = \underset{(1xr)}{?N(x, y, z)?} \underset{(rx1)}{?T(t)?}$$



FINITE ELEMENT EQUATIONS

Then the temperature gradients are,

$$\underset{(3x1)}{\begin{Bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{Bmatrix}} = \underbrace{\begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_r}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots & \frac{\partial N_r}{\partial y} \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots & \frac{\partial N_r}{\partial z} \end{bmatrix}}_{\underset{(3xr)}{[B(x,y,z)]}} \underset{(rx1)}{\{T(t)\}}$$

where $[B(x, y, z)]$ is the temperature gradient interpolation matrix.

FINITE ELEMENT EQUATIONS

Apply the Method of Weighted Residuals (MWR) with Bubnov-Galerkin technique,

$$\int_{\Omega^{(e)}} \left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} - Q + \rho c \frac{\partial T}{\partial t} \right] N_i \, d\Omega = 0$$

Expand,

$$\int_{\Omega^{(e)}} \left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right] N_i \, d\Omega - \int_{\Omega^{(e)}} Q \, N_i \, d\Omega + \int_{\Omega^{(e)}} \rho c \frac{\partial T}{\partial t} N_i \, d\Omega = 0$$

FINITE ELEMENT EQUATIONS

Perform integration by parts using Gauss's theorem to get,

$$\int_{\Gamma^{(e)}} (\vec{q}\cdot\hat{n})\, N_i \, d\Gamma \;-\; \int_{\Omega^{(e)}} \left[\frac{\partial N_i}{\partial x} \;\; \frac{\partial N_i}{\partial y} \;\; \frac{\partial N_i}{\partial z} \right] \begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} d\Omega$$

$$-\; \int_{\Omega^{(e)}} Q\, N_i \, d\Omega \;+\; \int_{\Omega^{(e)}} \rho c \frac{\partial T}{\partial t} N_i \, d\Omega \;=\; 0$$

Then use the Fourier's law to yield,

FINITE ELEMENT EQUATIONS

$$\int_{\Gamma^{(e)}} (\vec{q}\cdot\hat{n})\, N_i \, d\Gamma \;+\; \int_{\Omega^{(e)}} [B]^T [k][B]\, d\Omega \,\{T\} \;-\; \int_{\Omega^{(e)}} Q\{N\}\, d\Omega$$

$$+\; \int_{\Omega^{(e)}} \rho c \{N\}[N]\, d\Omega \begin{Bmatrix} \frac{\partial T}{\partial t} \end{Bmatrix} \;=\; 0$$

Finally, apply the boundary conditions on the boundary integral term leading to the finite element eqs. in the form,

$$[C]\{\dot{T}\} + \left([K_c] + [K_h] + [K_r]\right)\{T\}$$

$$= \{Q_c\} + \{Q_Q\} + \{Q_q\} + \{Q_h\} + \{Q_r\}$$

FINITE ELEMENT MATRICES

$$?C??\dot{T}? + ??K_c? + ?K_h? + ?K_r???T?$$

$$= ?Q_c? + ?Q_Q? + ?Q_q? + ?Q_h? + ?Q_r?$$

where

$$?C? = \int_{?^{(e)}} ?\ c ?N??N? d?$$

$$?K_c? = \int_{?^{(e)}} ?B?^T ?k??B? d?$$

$$?K_h? = \int_{S_3} h ?N??N? d?$$

$$?K_r? = \int_{S_4} ??\ T^3 ?N??N? d?$$

FINITE ELEMENT MATRICES

$$?Q_c? = -\int_{S_1} ?q.\hat{n}??N? d?$$

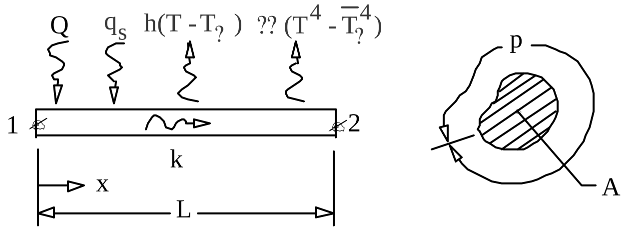
$$?Q_Q? = \int_{?^{(e)}} Q ?N? d?$$

$$?Q_q? = \int_{S_2} q_s ?N? d?$$

$$?Q_h? = \int_{S_3} h\ T_? ?N? d?$$

$$?Q_r? = \int_{S_4} ??? \overline{T}_?^4 ??\ q_r ??N? d?$$

ONE-DIMENSIONAL ELEMENT



Here $$T(x, y, t) = \underset{(1x2)}{\lfloor N(x) \rfloor} \underset{(2x1)}{\{T(t)\}} = \left\lfloor 1 - \frac{x}{L} \quad \frac{x}{L} \right\rfloor \begin{Bmatrix} T_1(t) \\ T_2(t) \end{Bmatrix}$$

Then $$\frac{\partial T}{\partial x} = \underset{(1x2)}{\left\lfloor \frac{\partial N_1}{\partial x} \quad \frac{\partial N_2}{\partial x} \right\rfloor} \underset{(2x1)}{\{T(t)\}} = \underset{\underset{(1x2)}{\lfloor B(x) \rfloor}}{\left\lfloor -\frac{1}{L} \quad \frac{1}{L} \right\rfloor} \underset{(2x1)}{\begin{Bmatrix} T_1(t) \\ T_2(t) \end{Bmatrix}}$$

ONE-DIMENSIONAL ELEMENT

$$[C] = \frac{\rho c A L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \{Q_Q\} = \frac{QAL}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

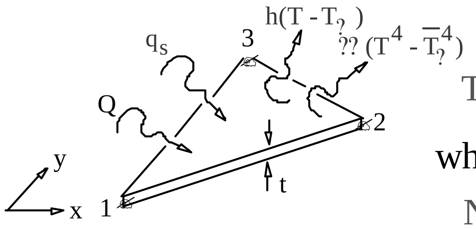
$$[K_c] = \frac{kA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \{Q_q\} = \frac{q_s pL}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$[K_h] = \frac{hpL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \{Q_h\} = \frac{hT_\infty pL}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\{Q_r\} = \left(\sigma\varepsilon \bar{T}_\infty^4 + \alpha q_r\right) \frac{pL}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \qquad \{Q_c\} = \begin{Bmatrix} -kA \frac{dT}{dx}(0) \\ kA \frac{dT}{dx}(L) \end{Bmatrix}$$

$$[K_r] = \frac{\varepsilon\sigma pL}{60} \begin{bmatrix} 10T_1^3 + 6T_1^2T_2 + 3T_1T_2^2 + T_2^3 & 2T_1^3 + 3T_1^2T_2 + 3T_1T_2^2 + 2T_2^3 \\ 2T_1^3 + 3T_1^2T_2 + 3T_1T_2^2 + 2T_2^3 & T_1^3 + 3T_1^2T_2 + 6T_1T_2^2 + 10T_2^3 \end{bmatrix}$$

TRIANGULAR ELEMENT



Element temperature,

$$T(x, y, t) = \underbrace{\lfloor N(x, y) \rfloor}_{(1x3)} \underbrace{\{T(t)\}}_{(3x1)}$$

where,

$$N_i = \frac{1}{2A}(a_i + b_i x + c_i y)$$

$$i = 1,\ 2,\ 3$$

Then the temperature gradients are,

$$\begin{Bmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \end{Bmatrix} = \underbrace{\frac{1}{2A}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}}_{\underset{(2x3)}{[B]}} \underbrace{\{T(t)\}}_{(3x1)}$$

TRIANGULAR ELEMENT

$$[C] = \frac{\rho c A t}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad [K_h] = \frac{hA}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$[K_c] = \frac{kt}{4A}\begin{bmatrix} b_1b_1 + c_1c_1 & b_1b_2 + c_1c_2 & b_1b_3 + c_1c_3 \\ & b_2b_2 + c_2c_2 & b_2b_3 + c_2c_3 \\ \text{Sym} & & b_3b_3 + c_3c_3 \end{bmatrix}$$

$$\{Q_Q\} = \frac{QAt}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}; \quad \{Q_q\} = \frac{q_s A}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}; \quad \{Q_h\} = \frac{hT_\infty A}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

TRIANGULAR ELEMENT

If there is convection heat transfer along the edge length ℓ between nodes 1 and 2, then the corresponding matrices are,

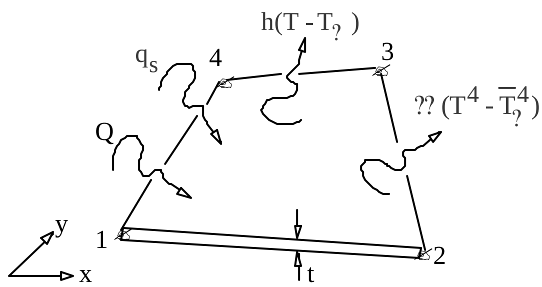
$$\left[K_h\right] = \frac{h\,t\,\ell}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left\{Q_h\right\} = \frac{h T_\infty\, t\,\ell}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

QUADRILATERAL ELEMENT

Element temperature,

$$T(\xi,\eta,t) = \underset{(1\times4)}{\lfloor N(\xi,\eta)\rfloor}\ \underset{(4\times1)}{\{T(t)\}}$$



Then the temperature gradients,

QUADRILATERAL ELEMENT

$$\begin{Bmatrix} \frac{?T}{?x} \\ \frac{?T}{?y} \end{Bmatrix} = \underbrace{\begin{bmatrix} J^{*}_{11} & J^{*}_{12} \\ J^{*}_{21} & J^{*}_{22} \end{bmatrix} \begin{bmatrix} \frac{?N_1}{??} & \frac{?N_2}{??} & \frac{?N_3}{??} & \frac{?N_4}{??} \\ \frac{?N_1}{??} & \frac{?N_2}{??} & \frac{?N_3}{??} & \frac{?N_4}{??} \end{bmatrix}}_{[B(?,?)]\ (2x4)} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix}$$

where J^{*}_{ij}, i, j = 1, 2 are the coefficients in the inverse Jacobian matrix.

QUADRILATERAL ELEMENT

Finite element matrices,

$$[C] = \int_{-1}^{1}\int_{-1}^{1} ?c\,\{N(?,?)\}\{N(?,?)\}^{?}\,t\,|J|\,d?\,d?$$

$$[K_c] = \int_{-1}^{1}\int_{-1}^{1} [B(?,?)]^{T}\,[k]\,[B(?,?)]\,t\,|J|\,d?\,d?$$

$$[K_h] = \int_{-1}^{1}\int_{-1}^{1} h\,\{N(?,?)\}\{N(?,?)\}^{?}\,|J|\,d?\,d?$$

$$\{Q_Q\} = \int_{-1}^{1}\int_{-1}^{1} Q\,\{N(?,?)\}\,t\,|J|\,d?\,d?$$

$$\{Q_q\} = \int_{-1}^{1}\int_{-1}^{1} q_S\,\{N(?,?)\}\,|J|\,d?\,d?$$

$$\{Q_h\} = \int_{-1}^{1}\int_{-1}^{1} h\,T_{?}\,\{N(?,?)\}\,|J|\,d?\,d?$$

QUADRILATERAL ELEMENT

These element matrices are evaluated using Gauss-Legendre numerical integration. For examples,

$$?C? = ?_{-1}^{1} ?_{-1}^{1} ? c ?N(?,?)? ?N(?,?)? t |J| d? d?$$

$$? \sum_{i?1}^{NG} \sum_{j?1}^{NG} W_i W_j ? c ?N(?_i,?_j)? ?N(?_i,?_j)? t |J(?_i,?_j)|$$

QUADRILATERAL ELEMENT

and

$$?K_c? = ?_{-1}^{1} ?_{-1}^{1} ?B(?,?)?^T ?k? ?B(?,?)? t |J| d? d?$$

$$= \sum_{i=1}^{NG} \sum_{j=1}^{NG} W_i W_j ?B(?_i,?_j)?^T ?k? ?B(?_i,?_j)? t |J(?_i,?_j)|$$

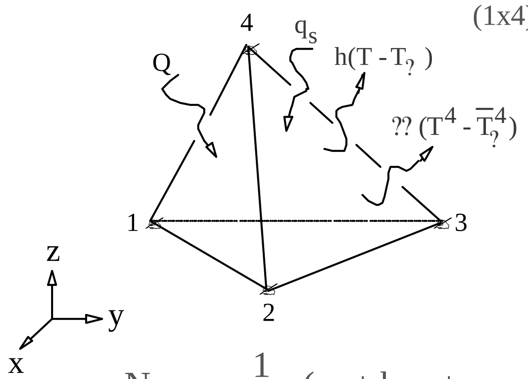
where W_i, W_j are the weights; $?_i$, $?_j$ are the Gauss point locations for the total of NG points.

TETRAHEDRAL ELEMENT

Element temperature,

$$T(x, y, z, t) = \lfloor N(x, y, z) \rfloor \{T(t)\}$$

$$\qquad\qquad\qquad\qquad (1x4) \qquad (4x1)$$



where the interpolation functions,

$$N_i = \frac{1}{6A}(a_i + b_i x + c_i y + d_i z) \qquad i = 1, 2, 3, 4$$

TETRAHEDRAL ELEMENT

Then the element temperature gradients are,

$$\begin{Bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{Bmatrix} = \frac{1}{6V} \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix} \{T(t)\}$$

$$[B] \; (3x4) \qquad \{T(t)\} \; (4x1)$$

TETRAHEDRAL ELEMENT

Element matrices can be derived in closed form. For examples,

$$[C] = \frac{\rho c V}{20}\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}; \qquad \{Q_Q\} = \frac{QV}{4}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$K_{c_{ij}} = \frac{1}{36V}\left(b_i b_j + c_i c_j + d_i d_j\right) \qquad i, j = 1, 2, 3, 4$$

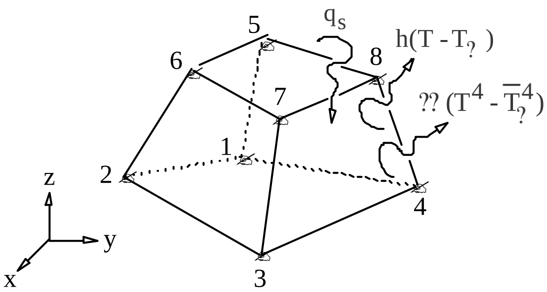
TETRAHEDRAL ELEMENT

If there is convection heat transfer on the element face connecting nodes 2-3-4, then the corresponding matrices are,

$$[K_h] = \frac{hA}{12}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}; \qquad \{Q_h\} = \frac{hT_\infty A}{3}\begin{Bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

where A is the area of that face.

HEXAHEDRAL ELEMENT



Element temperature distribution,

$$T(?, ?, ?, t) \quad = \quad ?N(?, ?, ?)? \; ?T(t)?$$
$$\qquad\qquad (1x8) \qquad (8x1)$$

where $N_i(?, ?, ?)$, $i = 1, 8$ are the element interpolation functions.

HEXAHEDRAL ELEMENT

The element temperature gradient can be derived,

$$\begin{Bmatrix} \frac{?T}{?x} \\ \frac{?T}{?y} \\ \frac{?T}{?z} \end{Bmatrix} = \begin{bmatrix} J^?_{11} & J^?_{12} & J^?_{13} \\ J^?_{21} & J^?_{22} & J^?_{23} \\ J^?_{31} & J^?_{32} & J^?_{33} \end{bmatrix} \begin{bmatrix} \frac{?N_1}{??} & \frac{?N_2}{??} & \cdots & \frac{?N_8}{??} \\ \frac{?N_1}{??} & \frac{?N_2}{??} & \cdots & \frac{?N_8}{??} \\ \frac{?N_1}{??} & \frac{?N_2}{??} & \cdots & \frac{?N_8}{??} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ \vdots \\ T_8 \end{Bmatrix}$$

$$= \; ?B(?, ?, ?)? \; ?T(t)?$$
$$\qquad (3x8) \qquad (8x1)$$

where $J^?_{ij}$, $i, j = 1, 2, 3$ are the coefficients in the inverse Jacobian matrix and $?B(?, ?, ?)?$ is the temperature gradient interpolation matrix needed in the derivation of the conduction matrix.

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HEXAHEDRAL ELEMENT

Then finite element matrices can be derived in integral form. Typical element matrices, such as the capacitance and conduction matrices, including load vector from heat generation are,

$$[C] = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \rho c \{N(\xi,\eta,\zeta)\}\{N(\xi,\eta,\zeta)\}^T |J| \, d\xi \, d\eta \, d\zeta$$

$$[K_c] = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} [B(\xi,\eta,\zeta)]^T [k] [B(\xi,\eta,\zeta)] |J| \, d\xi \, d\eta \, d\zeta$$

$$\{Q_Q\} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} Q \{N(\xi,\eta,\zeta)\} |J| \, d\xi \, d\eta \, d\zeta$$

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HEXAHEDRAL ELEMENT

These element matrices are evaluated using Gauss-Legendre numerical integration as,

$$[C] = \sum_{i=1}^{NG}\sum_{j=1}^{NG}\sum_{k=1}^{NG} W_i W_j W_k \, \rho c \{N(\xi_i,\eta_j,\zeta_k)\}\{N(\xi_i,\eta_j,\zeta_k)\}^T |J(\xi_i,\eta_j,\zeta_k)|$$

$$[K_c] = \sum_{i=1}^{NG}\sum_{j=1}^{NG}\sum_{k=1}^{NG} W_i W_j W_k [B(\xi_i,\eta_j,\zeta_k)]^T [k] [B(\xi_i,\eta_j,\zeta_k)] |J(\xi_i,\eta_j,\zeta_k)|$$

$$\{Q_Q\} = \sum_{i=1}^{NG}\sum_{j=1}^{NG}\sum_{k=1}^{NG} W_i W_j W_k Q \{N(\xi_i,\eta_j,\zeta_k)\} |J(\xi_i,\eta_j,\zeta_k)|$$

where W_i, W_j, W_k are the weights; ξ_i, η_j, ζ_k are the Gauss point locations for the total of NG points.

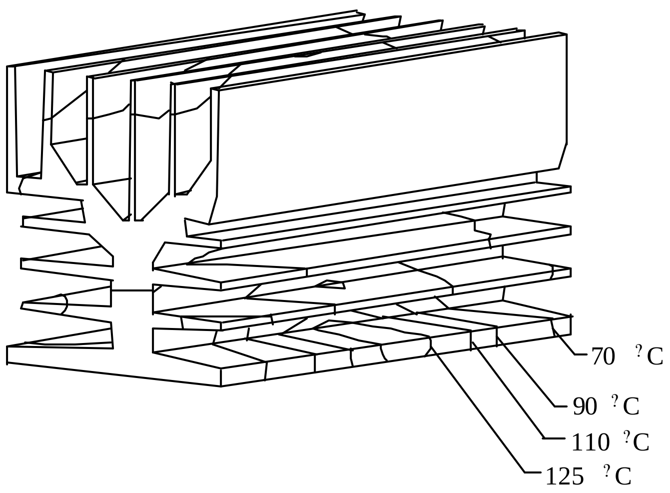
LINEAR STEADY-STATE HEAT TRANSFER

For linear steady-state heat transfer problems, the derived finite element equations reduce to,

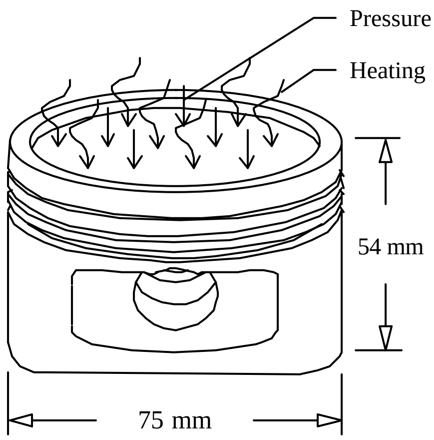
$$\left([K_c] + [K_h] \right) \{T\} = \{Q_c\} + \{Q_Q\} + \{Q_q\} + \{Q_h\}$$

Solving such problems poses no difficulty. The system of algebraic eqs. can be solved conveniently using standard methods such as Gauss elimination, LU decomposition, etc.

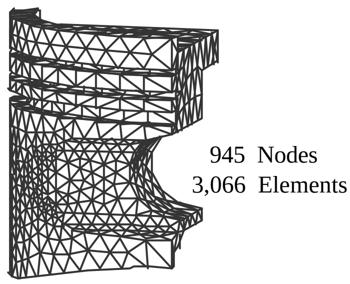
AMPLIFIER FIN TEMPERATURE



MOTORCYCLE PISTON

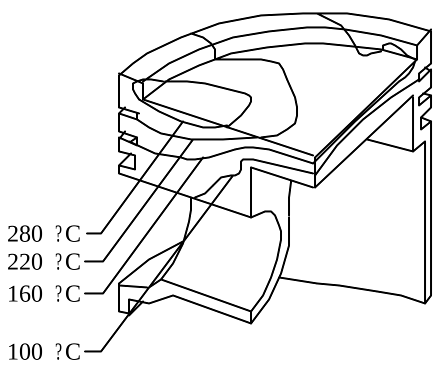


F. E. Model

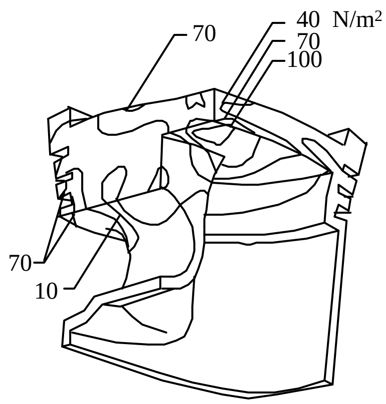


MOTORCYCLE PISTON

Temperature



Von Mises Stress



LINEAR TRANSIENT HEAT TRANSFER

Finite element equations:

$$[C]\{\dot{T}\} + ([K_c] + [K_h])\{T\} = \{Q_c\} + \{Q_Q\} + \{Q_q\} + \{Q_h\}$$

The square matrices on LHS are not function of temperature and the load vectors on RHS may depend on time. These element eqs. can be written, in short, as

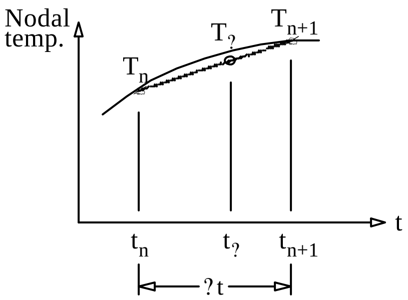
$$[C]\{\dot{T}\} + [K]\{T\} = \{Q\}$$

where

$$[K] = [K_c] + [K_h]$$

$$\{Q\} = \{Q_c\} + \{Q_Q\} + \{Q_q\} + \{Q_h\}$$

RECURRENCE RELATIONS



Knowing temperature T_n at time t_n, want to compute temperature T_{n+1} at time t_{n+1} using the time step of ? t.

From the figure,

$$t_? = t_n + ? ? t \qquad 0 ? ? ? 1$$

and the approximate temperature gradient is,

$$\dot{T}_? \; ? \; \frac{T_{n+1} - T_n}{? \, t}$$

Then, the temperature at time $t_?$ is,

$$T_? \; ? \; (1-?)\,T_n + ?\,T_{n+1}$$

RECURRENCE RELATIONS

Finite element equations are evaluated at time $t_?$,

$$?C??\dot{T}?_? + ?K??T?_? = ?Q?_?$$

Since the nodal temperature gradients are,

$$?\dot{T}?_? \quad ? \quad \frac{?T?_{n+1} - ?T?_n}{? \ t}$$

and the nodal temperatures,

$$?T?_? \quad ? \quad (1-?)?T?_n + ??T?_{n+1}$$

Also the load vector at time $t_?$,

$$?Q?_? \quad ? \quad (1-?)?Q?_n + ??Q?_{n+1}$$

Substitute these into the FE eqs. above to yield,

RECURRENCE RELATIONS

$$?\frac{1}{??t}?C?? ?K???T?_{n+1} = ?\frac{1}{??t}?C?-(1-?)?K???T?_n$$

$$+ (1-?)?Q?_n + ??Q?_{n+1}$$

Solution procedure and results depend on ? selected,

??	Method
0	Euler
1/2	Crank-Nicolson
2/3	Galerkin
1	Backward difference

EULER METHOD

If $\theta = 0$, the element equations reduce to,

$$\frac{1}{\Delta t}[C]\{T\}_{n+1} = \left(\frac{1}{\Delta t}[C] - [K]\right)\{T\}_n + \{Q\}_n$$

In addition, if [C] is lumped, then can solve "<u>Uncoupled equations</u>". As an example, the capacitance matrix for 1-D rod element is,

$$[C] = \rho c A L \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

EULER METHOD

The corresponding lumped capacitance matrix is,

$$[C]_{lumped} = \rho c A L \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

For other element types, the coefficients along the diagonal line of the lumped capacitance matrix are determined from,

$$C_{ii} = \int_{\Omega^{(e)}} \rho c N_i \, d\Omega$$

EULER METHOD

Note that the Euler method,

- leads to uncoupled eqs. called "Explicit method", that helps saving computer memory.
- the time step used must be less than the critical time step, if not, the solution can diverge.

CRANK-NICOLSON METHOD

If $\theta = 1/2$, the element equations become,

$$\left[\frac{1}{\Delta t}[C] + \frac{1}{2}[K]\right]\{T\}_{n+1} = \left[\frac{1}{\Delta t}[C] - \frac{1}{2}[K]\right]\{T\}_{n} + \frac{1}{2}\left(\{Q\}_{n} + \{Q\}_{n+1}\right)$$

which are coupled equations. The method is called "Implicit method" that,

- can use larger time step than the Euler method
- can provide higher solution accuracy
- may give oscillation of solution if time step is too high

GALERKIN & BACKWARD DIFFERENCE METHODS

$$\left[\frac{1}{\Delta t}[C] + \theta[K]\right]\{T\}_{n+1} = \left[\frac{1}{\Delta t}[C] - (1-\theta)[K]\right]\{T\}_n + (1-\theta)\{Q\}_n + \theta\{Q\}_{n+1}$$

<u>Galerkin method</u> ($\theta = 2/3$)

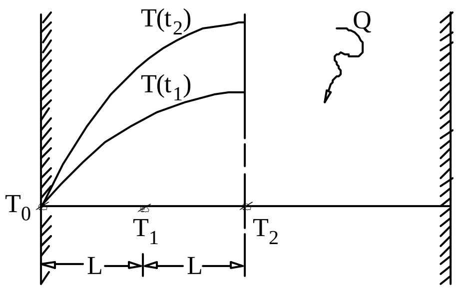
- If use large time step, oscillation is less than Crank-Nicolson
- However, solution accuracy is also reduced

<u>Backward difference method</u> ($\theta = 1$)

- Provides smooth solution with no oscillation
- Gives less solution accuracy compared to other methods

TRANSIENT HEAT CONDUCTION IN SLAB

Governing diff. eq.: $\rho c \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} = Q$



Boundary conditions,

$T(0,t) = T(L,t) = 0$

Initial condition

$T(x,0) = 0$ and

$$Q(x,t) = \begin{cases} 0 & t < 0 \\ Q_0 & t \geq 0 \end{cases}$$

Use 2 elements to compute transient nodal temperatures $T_1(t)$ and $T_2(t)$ in the figure.

TRANSIENT HEAT CONDUCTION IN SLAB

Finite element eqs. for this problem are,

$$[C]\{\dot{T}\} + [K_c]\{T\} = \{Q_c\} + \{Q_Q\}$$

and the typical element eqs. are,

$$\rho cAL \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{Bmatrix} \dot{T}_1 \\ \dot{T}_2 \end{Bmatrix} + \frac{kA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} -kA\frac{dT}{dx}(0) \\ kA\frac{dT}{dx}(L) \end{Bmatrix} + \frac{Q_0 AL}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

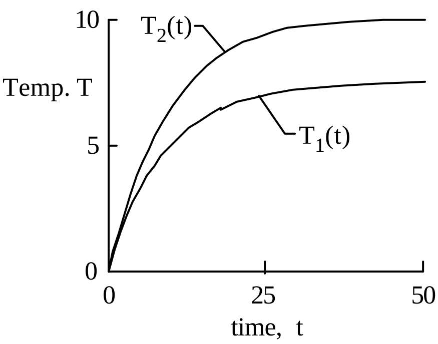
TRANSIENT HEAT CONDUCTION IN SLAB

Given $L = 1/4$, $\rho = 4$, $c = 3$, $k = 1/8$ and $Q_0 = 10$ element eqs. are assembled. After applying boundary conditions, the system eqs. reduce to,

$$\begin{bmatrix} 2 & 0.5 \\ 0.5 & 1 \end{bmatrix} \begin{Bmatrix} \dot{T}_1 \\ \dot{T}_2 \end{Bmatrix} + \begin{bmatrix} 1 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 2.50 \\ 1.25 \end{Bmatrix}$$

TRANSIENT HEAT CONDUCTION IN SLAB

Then applying the Crank-Nicolson method with $\Delta t = 1$, this leads to the solution below,



n	t	$T_1(t)$	$T_2(t)$
0	0.	0.00	0.00
1	1.	0.92	0.82
2	2.	1.62	1.72
3	3.	2.23	2.56
4	4.	2.78	3.32
5	5.	3.26	4.00
⋮	⋮	⋮	⋮
50	50.	7.47	9.95

TRANSIENT HEAT CONDUCTION IN SLAB

If use the Euler method, the system eqs. are uncoupled,

$$\begin{bmatrix} 3 & 0 \\ 0 & 1.5 \end{bmatrix} \begin{Bmatrix} \dot{T}_1 \\ \dot{T}_2 \end{Bmatrix} + \begin{bmatrix} 1 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 2.50 \\ 1.25 \end{Bmatrix}$$

		Euler		Crank-Nicolson	
n	t	$T_1(t)$	$T_2(t)$	$T_1(t)$	$T_2(t)$
0	0.	0.00	0.00	0.00	0.00
1	1.	0.83	0.83	0.92	0.82
2	2.	1.53	1.67	1.62	1.72
3	3.	2.13	2.45	2.23	2.56
4	4.	2.66	3.18	2.78	3.32
5	5.	3.14	3.84	3.26	4.00
⋮	⋮	⋮	⋮	⋮	⋮
50	50.	7.46	9.94	7.47	9.95

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NONLINEAR STEADY-STATE HEAT TRANSFER

Finite element equations:

$$\left([K_c(T)] + [K_h(T)] + [K_r(T)] \right) \{T\} = \{Q\}$$

As an example, the conduction matrix

$$[K_c] = \int_\Omega [B]^T [k(T)] [B] \, d\Omega$$

The above element eqs. can be written, in short, as:

$$[K(T)]\{T\} = \{Q\}$$

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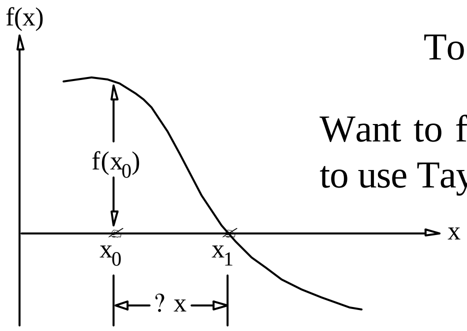
NEWTON-RAPHSON ITERATION TECHNIQUE



To solve, $f(x) = 0$

Want to find the root x_1. Idea is to use Taylor series expansion,

$$f(x_1) = f(x_0 + \Delta x) = f(x_0) + \left.\frac{df}{dx}\right|_{x_0} \Delta x + \ldots\ldots = 0$$

NEWTON-RAPHSON ITERATION TECHNIQUE

Compute $?x$ by taking only the first two terms,

$$\underbrace{\left.\frac{df}{dx}\right|_{x_0}}_{J(x^0)} \; ?x \;=\; -f(x_0)$$

Then the approximate x_1 is,

$$x_1 = x_0 + ?x$$

NEWTON-RAPHSON ITERATION TECHNIQUE

In conclusion, the iteration procedure is,

1. Solve $J(x^m)\,?x^{m+1} = -f(x^m)$

   where $J(x^m)$ is the Jacobian at the m^{th} iteration
2. Update $x^{m+1} = x^m + ?x^{m+1}$
3. Check for convergence. If yes ⇨ Stop

   No ⇨ Go to step 1

NEWTON-RAPHSON ITERATION TECHNIQUE

Now, for a set of n simultaneous finite element eqs., start from guessing a set of nodal temperature solutions,

$$?K(T)??T? - ?Q? \quad ? \quad 0 \quad \text{but} \quad = \quad ?R?$$

where $?R?$ is the residual vector. The residual for the i^{th} eq. is

$$R_i \quad = \quad \overset{n}{\underset{j=1}{?}} K_{ij} (T_1, T_2, ..., T_n) T_j - Q_i$$

NEWTON-RAPHSON ITERATION TECHNIQUE

Similary, apply Taylor series expansion,

$$R_i (T_1 + ? T_1, T_2 + ? T_2, ..., T_n + ? T_n)$$

$$= R_i (T_1, T_2, ..., T_n) + \overset{n}{\underset{j?1}{?}} \frac{?R_i}{?T_j} (T_1, T_2, ..., T_n) ? T_j + ...$$

$$= 0$$

and take only the first two terms to get,

$$\overset{n}{\underset{j?1}{?}} \frac{?R_i}{?T_j} (T_1, T_2, ..., T_n) ? T_j \quad = \quad - R_i (T_1, T_2, ..., T_n)$$

NEWTON-RAPHSON ITERATION TECHNIQUE

which can be written in matrix form as,

$$\underbrace{\begin{bmatrix} \frac{\partial R_1}{\partial T_1} & \frac{\partial R_1}{\partial T_2} & \cdots & \frac{\partial R_1}{\partial T_n} \\ \frac{\partial R_2}{\partial T_1} & \frac{\partial R_2}{\partial T_2} & \cdots & \frac{\partial R_2}{\partial T_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial R_n}{\partial T_1} & \frac{\partial R_n}{\partial T_2} & \cdots & \frac{\partial R_n}{\partial T_n} \end{bmatrix}}_{\substack{[J] \\ (nxn)}} \underbrace{\begin{Bmatrix} \Delta T_1 \\ \Delta T_2 \\ \vdots \\ \Delta T_n \end{Bmatrix}}_{\substack{\{\Delta T\} \\ (nx1)}} = - \underbrace{\begin{Bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{Bmatrix}}_{\substack{\{R\} \\ (nx1)}}$$

NEWTON-RAPHSON ITERATION TECHNIQUE

Thus the iteration procedure is,

1. Solve $\quad [J(T)]^m \{\Delta T\}^{m+1} = -\{R\}^m$

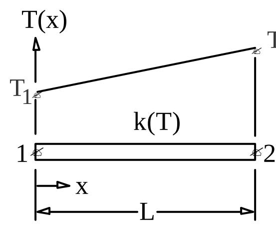
   where m is the m^{th} iteration.

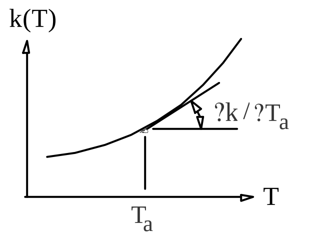
2. Update $\quad \{T\}^{m+1} = \{T\}^m + \{\Delta T\}^{m+1}$

   and check for convergence.

TEMPERATURE DEPENDENT MATERIAL PROPERTIES

<u>Example</u> Derive the Jacobian matrix for rod element with temperature dependent thermal conductivity.





Here, the element eqs. are, $[K(T)]\{T\} = \{Q\}$

Then the residuals vector is, $\{R\} = [K(T)]\{T\} - \{Q\}$

TEMPERATURE DEPENDENT THERMAL CONDUCTIVITY

Since the coefficients in the Jacobian matrix are,

$$J_{ij} = \frac{\partial R_i}{\partial T_j} = \frac{\partial}{\partial T_j} \sum_{\ell=1}^{2} (K_{i\ell} T_\ell - Q_i)$$

$$= \sum_{\ell=1}^{2} K_{i\ell} \frac{\partial T_\ell}{\partial T_j} + \sum_{\ell=1}^{2} \frac{\partial K_{i\ell}}{\partial T_j} T_\ell - 0$$

$$J_{ij} = K_{ij} + \sum_{\ell=1}^{2} \frac{\partial K_{i\ell}}{\partial T_j} T_\ell$$

TEMPERATURE DEPENDENT THERMAL CONDUCTIVITY

Since the average temperature is $T_a = (T_1 + T_2)/2$, then,

$$J_{ij} = K_{ij} + \sum_{\ell=1}^{2} \frac{\partial K_{i\ell}}{\partial T_a} \frac{\partial T_a}{\partial T_j} T_\ell$$

$$= K_{ij} + \frac{1}{2} \sum_{\ell=1}^{2} \frac{\partial K_{i\ell}}{\partial T_a} T_\ell$$

TEMPERATURE DEPENDENT THERMAL CONDUCTIVITY

Thus, the Jacobian matrix is,

$$[J] = \frac{kA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{1}{2}\frac{\partial k}{\partial T_a}\frac{A}{L}\begin{bmatrix} T_1 - T_2 & T_1 - T_2 \\ -T_1 + T_2 & -T_1 + T_2 \end{bmatrix}$$

which can be used for programming directly. Note that $\partial k/\partial T_a$ is the slope of thermal conductivity wrt. temperature at the average element temperature. Such quantity can be computed from user input table. Also note that [J] is unsymmetric matrix, thus more memory is required in addition to more CPU time from iteration procedure.

NONLINEARLITY FROM RADIATION

Finite element equations are,

$$\left(\left[K_c\right] + \left[K_r(T)\right]\right)\left\{T\right\} = \left\{Q_c\right\} + \left\{Q_r\right\}$$

From a set of guessing nodal temperatures, the residual vector is,

$$\left\{R\right\} = \underbrace{\left[K_c\right]\left\{T\right\}}_{\left\{R\right\}_{cond}} + \underbrace{\int_S \varepsilon\sigma T^4 \left\{N\right\} d\Gamma}_{\left\{R\right\}_{rad}} - \underbrace{\left\{Q_c\right\} - \left\{Q_r\right\}}_{\left\{R\right\}_{load}}$$

NONLINEARLITY FROM RADIATION

Applying the Newton-Raphson iteration method,

$$\left[J\right]^m \left\{\Delta T\right\}^{m+1} = -\left\{R\right\}^m$$

Here the coefficients in the Jacobian matrix are,

$$J_{ij} = (J_{ij})_{cond} + (J_{ij})_{rad} + (J_{ij})_{load}$$

$$= \left(\frac{\partial R_i}{\partial T_j}\right)_{cond} + \left(\frac{\partial R_i}{\partial T_j}\right)_{rad} + \left(\frac{\partial R_i}{\partial T_j}\right)_{load}$$

RADIATION HEAT TRANSFER

For independent material thermal conductivity,

$$(J_{ij})_{cond} = K_{c_{ij}} \quad \text{or} \quad ?J?_{cond} = ?K_c?$$

The Jacobian matrix coefficients from radiation can be derived,

$$(J_{ij})_{rad} = \frac{?}{?T_j}\left(?_{S_4} ?? \, T^4 N_i \, d? \right)$$

$$= ?_{S_4} \left(?? \left(4T^3 \frac{?T}{?T_j}\right) N_i \, d? \right)$$

But $\frac{?T}{?T_j} = \frac{?}{?T_j}(N_{□}T_{□}) = N_{□}?_{□j} = N_j$

RADIATION HEAT TRANSFER

Then $(J_{ij})_{rad} = ?_{S_4} 4?? \, T^3 N_i N_j \, d?$ or

$$?J?_{rad} = ?_{S_4} 4?? \, T^3 ?N??N? d?$$

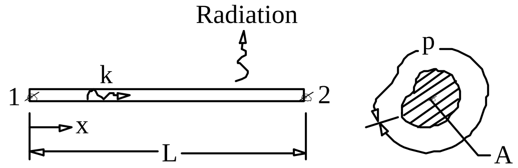
Thus, the finite element eqs. needed for solution are,

$$??J?^m_{cond} ? ?J?^m_{rad}??T?^{m+1} = -?R?^m_{cond} - ?R?^m_{rad} - ?R?^m_{load}$$

where m is the m^{th} iteration.

RADIATION HEAT TRANSFER

Example Derive the Jacobian matrices and residual vectors for rod element with surface radiation.



Here $T(x) = \lfloor N_1 \quad N_2 \rfloor \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \left\lfloor 1 - \frac{x}{L} \quad \frac{x}{L} \right\rfloor \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$

Then

$$[J]_{cond} = [K_c] = \int_0^L \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} k \lfloor N_1 \quad N_2 \rfloor (A\,dx) = \frac{kA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

ROD WITH SURFACE RADIATION

$$[J]_{rad} = \int_0^L 4 \, ? \, ? \, (N_1 T_1 + N_2 T_2)^3 \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} \lfloor N_1 \quad N_2 \rfloor (p\,dx)$$

$$= \frac{??pL}{15} \begin{bmatrix} 10T_1^3 \,?\, 6T_1^2T_2 \,?\, 3T_1T_2^2 \,?\, T_2^3 & 2T_1^3 \,?\, 3T_1^2T_2 \,?\, 3T_1T_2^2 \,?\, 2T_2^3 \\ 2T_1^2 \,?\, 3T_1^2T_2 \,?\, 3T_1T_2^2 \,?\, 2T_2^3 & T_1^3 \,?\, 3T_1^2T_2 \,?\, 6T_1T_2^2 \,?\, 10T_2^3 \end{bmatrix}$$

$$\{R\}_{cond} = [K_c]\{T\} = \frac{kA}{L} \begin{Bmatrix} T_1 - T_2 \\ -T_1 + T_2 \end{Bmatrix}$$

$$\{R\}_{rad} = \int_0^L ? \, ? \, (N_1 T_1 + N_2 T_2)^4 \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} (p\,dx)$$

$$= \frac{??pL}{30} \begin{Bmatrix} 5T_1^4 \,?\, 4T_1^3T_2 \,?\, 3T_1^2T_2^2 \,?\, 2T_1T_2^3 \,?\, T_2^4 \\ T_1^4 \,?\, 2T_1^3T_2 \,?\, 3T_1^2T_2^2 \,?\, 4T_1T_2^3 \,?\, 5T_2^4 \end{Bmatrix}$$

NONLINEAR TRANSIENT HEAT TRANSFER

______Finite element equations,

$$[C(T)]\{\dot{T}(t)\} + [K(T)]\{T(t)\} = \{Q(T,t)\}$$

<u>Example</u> Derive the Jacobian and the residual for a single DOF nonlinear transient equation,

$$\dot{T} + (a + bT)\,T = 0$$

where a and b are constants. The initial condition is $T(t{=}0) = T_0$.

NONLINEAR TRANSIENT HEAT TRANSFER

______To solve this problem, we need to apply the recurrence relations. and then the Newton-Raphson iteration. By comparing the given diff. eq. and the FE eqs., we have,

$$[C(T)] = 1, \qquad [K(T)] = a + bT, \qquad \{Q(T, t)\} = 0$$

As an example, applying Crank-Nicolson (θ = 1/2) leads to,

$$\left(\frac{1}{\Delta t} + \frac{1}{2}(a + bT_{n+1/2})\right)T_{n+1} = \left(\frac{1}{\Delta t} - \frac{1}{2}(a + bT_{n+1/2})\right)T_n$$

where n is the n^{th} time step.

NONLINEAR TRANSIENT HEAT TRANSFER

Since $T_{n?1/2} = \frac{1}{2}(T_n + T_{n+1})$, then,

$$?1 + a\frac{?t}{2} ? b\frac{?t}{4}T_n ? b\frac{?t}{4}T_{n?1}? T_{n+1}$$

$$= ?1 - a\frac{?t}{2} - b\frac{?t}{4}T_n - b\frac{?t}{4}T_{n?1}? T_n$$

which is in form of nonlinear equation,

$$\overline{K}(T_{n+1})\,T_{n+1} = \overline{Q}(T_{n+1})$$

NONLINEAR TRANSIENT HEAT TRANSFER

Then applying the Newton-Raphson iteration method. The residual,

$$R = \overline{K}(T_{n+1})\,T_{n+1} - \overline{Q}(T_{n+1})$$

which leads to the incremental equation,

$$J^m \; ?T_{n+1}^{m+1} = -R^m$$

where m is the m^{th} iteration. Here,

$$R^m = ?1 + a\frac{?t}{2} ? b\frac{?t}{4}T_n ? b\frac{?t}{4}T_{n?1}^m ?T_{n+1}^m$$

$$- ?1 - a\frac{?t}{2} - b\frac{?t}{4}T_n - b\frac{?t}{4}T_{n?1}^m ?T_n$$

NONLINEAR TRANSIENT HEAT TRANSFER

Then the Jacobian is,

$$J^m = \frac{?R^m}{?T^m_{n?1}}$$

$$= 1 + a\frac{?t}{2} + b\frac{?t}{4}T_n + b\frac{?t}{4}T^m_{n+1}$$

In conclusion, at each time step, need to perform iteration and solve the incremental equation. The iteration process is repeated for the total number of time steps specified.

CONVECTIVE-DIFFUSION EQUATION

Governing differential equation,

$$- (\frac{?q_x}{?x} + \frac{?q_y}{?y} + \frac{?q_z}{?z}) - ?c(u\frac{?T}{?x} + v\frac{?T}{?y} + w\frac{?T}{?z}) + Q$$

$$= ?c\frac{?T}{?t}$$

where u, v, w are the given velocity components in the x, y, z directions, respectively. Using MWR, the corresponding FE eqs. can be derived as,

CONVECTIVE-DIFFUSION EQUATION

$$[C]\{\dot{T}\} + \left([K_c] + [K_h] + [K_v]\right)\{T\}$$

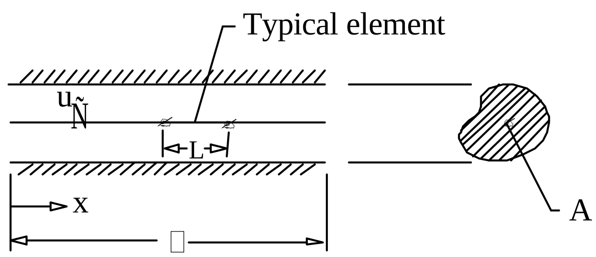
$$= \{Q_c\} + \{Q_Q\} + \{Q_q\} + \{Q_h\}$$

All matrices are the same as before, except the new matrix,

$$[K_v] = \int_{\Omega^{(e)}} \rho c \{N\} \lfloor u \quad v \quad w \rfloor [B] \, d\Omega$$

which is called "the mass transport convection matrix".

ONE-DIMENSIONAL STEADY-STATE CONVECTIVE-DIFFUSION EQUATION



Governing differential equation,

$$kA \frac{d^2T}{dx^2} - \rho c A u \frac{dT}{dx} = 0$$

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ONE-DIMENSIONAL STEADY-STATE CONVECTIVE-DIFFUSION EQUATION

and the corresponding finite element eqs. are,

$$\left(\left[K_c\right] + \left[K_v\right]\right)\{T\} = \{Q_c\}$$

If use standard linear element,

$$T(x) = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

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FINITE ELEMENT MATRICES

Then the conduction matrix and conduction vector are,

$$[K_c] = \int_0^L [B]^T k [B] (A\,dx) = \frac{kA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{Q_c\} = \left(\{N\}\,kA\frac{dT}{dx}\right)\Big|_0^L = \begin{Bmatrix} -kA\frac{dT}{dx}(0) \\ kA\frac{dT}{dx}(L) \end{Bmatrix}$$

and the mass transport convection matrix (Bubnov-Galerkin) is,

$$[K_v] = \int_0^L \rho c \{N\} u [B] (A\,dx) = \frac{\rho c A u}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

FINITE ELEMENT MATRICES

Then the finite element eqs. are,

$$\left(\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \frac{Pe}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\right)\begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = \begin{Bmatrix}-L\dfrac{dT}{dx}(0)\\ L\dfrac{dT}{dx}(L)\end{Bmatrix}$$

where $Pe = ?cuL/k$ is called "grid Peclet number" that depends on the element length.

PETROV-GALERKIN FORMULATION

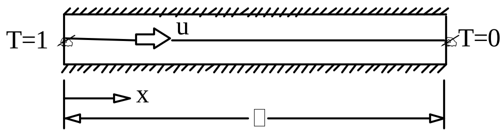
Note that solution oscillates if element length is too large. Oscillation can be eliminated by using different weighting functions,

$$\{W\} = \begin{Bmatrix}W_1\\ W_2\end{Bmatrix} = \begin{Bmatrix}N_1\\ N_2\end{Bmatrix} + 3?\left(\frac{x^2}{L^2} - \frac{x}{L}\right)\begin{Bmatrix}1\\ -1\end{Bmatrix}$$

where ? is a constant. If $? = 0$ then $\{W\}=\{N\}$ which is the standard Bubnov-Galerkin formulation. If $? = 1$ is called full upwinding finite element that always give smooth temperature distribution. The corresponding mass transport convection matrix is,

$$[K_v] = \int_0^L ?c\{W\}u[B](Adx) = \frac{?cAu}{2}\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix} + ?\,\frac{?cAu}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

CONVECTIVE-DIFFUSION EXAMPLE



Exact temperature distribution,

$$T(x) = 1 - (1 - e^{Pe\ x/L})/(1 - e^{Pe\ \ell/L})$$